On Curves of Descent

Alexander Ioffe
Israel Institute of Technology

In the classical analysis the curves of gradient (steepest) descent is an important instrument both in theory and practice, especially in areas connected with critical points and optimization.
In 1980 DeGiorgi, Marino and Tosques introduced the concept of slope that enabled them to extend the concept of steepest descent to non-differentiable functions. A number of existence theorems for "curves of maximal slope" have been proved following this innovation. In all of them certain assumption with a flavor of convexity have been imposed that effectively guarantee that the slope is a lower semi-continuous function. However, this is not the case even for the simplest non-convex and non-differentiable functions.

In the talk, based on a joint work with D. Drusviatskiy and A. Lewis, I shall introduce a concept of a "near-steepest decent" curve and explain a geometrically transparent idea behind the proof of the existence theorem for "curves of near-maximal slope". This proof can be applied already for all "not very discontinuous" lower semi-continuous functions but the main attention will be paid to functions with special structures that excludes pathologies (that usually do not appear in practice). The most important among them is the class of semi-algebraic functions, simply defined and convenient to work with.
Such functions always admit nontrivial descent curves emanating from any (even critical) non-minimizing point.
Moreover, it is shown that any curve of near-maximal slope of a semi-algebraic functions (a) have a more classical description as a solution of the subgradient dynamical systems and (b) has bounded length if the function is bounded below and the curve lies in a bounded region.