Arrovian Social Choice with Non -Welfare Attributes*

Ryo-Ichi Nagahisa†
Kansai University

Koichi Suga‡
Waseda University

July 31, 2017

Abstract

1 Introduction

2 Notation and Definitions

Let $N = \{1, 2, ..., n\}$ be the finite set of persons with at least two. Let X be the finite set of social states with at least three. Let $\succcurlyeq_i$ be the preference of person i. We assume that $\succcurlyeq_i$ is complete and transitive on X[1]. The strict and indifferent preferences associated with $\succcurlyeq_i$ are denoted by $\succ_i$ and $\sim_i$ respectively. Let $P(X)$ be the set of all preferences. A profile $\succcurlyeq$ is the list of individual preferences $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$, so the set of profiles is $P(X)^n$. A social choice rule F, simply a rule, is a mapping that associates with each profile $\succcurlyeq \in P(X)^n$ a social preference $\succcurlyeq_F$, a complete binary relation on X. The strict and indifferent social preference associated with $\succcurlyeq_F$ are denoted by $\succ_F$ and $\sim_F$ respectively.

A social state $x \in X$ is characterized by welfare and non-welfare attributes. Given a profile, the welfare attribute of x is the profile itself and all the concepts derived from the profile such as utilities of x, the Borda numbers of x and so on, which depend on profiles. On the other hand non-welfare attributes are intrinsic to x independently from profiles. Let non-welfare attributes be given. We assume all the social states are classified into subgroups in which each member is thought of as identical from the viewpoint of the non-welfare attributes. Thus

*This research was financially supported by the Kansai University Fund for Domestic Research Support, 2017.

†Department of Economics, 3-3-35 Yamatecho Suita 564-8680 Japan

‡The School of Political Science and Economics, 1-6-1 Nishi-Waseda Shinjuku Tokyo 169-8050 Japan

[1]We say $\succcurlyeq_i$ is complete on X if and only if for all $x, y \in X$, $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$, and $\succcurlyeq_i$ is transitive on X if and only if for all $x, y, z \in X$, $x \succcurlyeq_i y \succcurlyeq_i z$ implies $x \succcurlyeq_i z$.

X has a partition $\{X_\lambda\}_{\lambda\in\Lambda}$, i.e., $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ and $X_\lambda \cap X_{\lambda'} = \emptyset$ for all $\lambda \neq \lambda'$. If $x, y \in X_\lambda$, we cannot distinguish between x and y from the viewpoint of the non-welfare attribute. We call X_λ an attribute set. We assume that there exist at least two attribute sets.

Example 1 Lady Chatterley's Lover (Sen 1969)
$X = \{r_{AB}, r_A, r_B, r_0\}$.
The non-welfare attribute: Read or not, a kind of morality
The attribute sets: either $\{r_{AB}\}, \{r_A, r_B\}, \{r_0\}$ or $\{r_{AB}, r_A, r_B\}, \{r_0\}$.

Example 2 Marriage (Gibbard 1974)
$X = \{w_E, w_J, w_o\}$.
The non-welfare attribute: Marriage or not, one of customs
The attribute sets:$\{w_E, w_J\}, \{w_o\}$.

Example 3 Mac or Windows
$X = \{(m, m), (m, w), (w, m), (w, w)\}$.
The non-welfare attribute: Corporate or not.
The attribute sets:$\{(m, m), (w, w)\}, \{(m, w), (w, m)\}$.

Example 4 Building a commercial complex (C) or protecting natural environment (E)
$X = \{C, E\} \times \prod_{i=1}^{n} X_i$.
The non-welfare attribute: Environment.
The attribute sets: $\{C\} \times \prod_{i=1}^{n} X_i, \{E\} \times \prod_{i=1}^{n} X_i$.

3 Axioms

A rule F satisfies Conditional Full Rationality (CFR) if for any $\succcurlyeq \in P(X)^n$, any $X_\lambda, X_{\lambda'}, \lambda \neq \lambda'$ and any $\{x, y, z\} \subset X_\lambda \cup X_{\lambda'}$, $x \succcurlyeq_F y \succcurlyeq_F z$ implies $x \succcurlyeq_F z$. Note that transitivity of $\succcurlyeq_F$ does not always hold on $\{x, y, z\}$ if each of the three belongs to a different attribute set. It is easy to see that a rule F satisfies CFR if and only if for any $X_\lambda, X_{\lambda'}, \lambda \neq \lambda'$ and any $\{x, y, z\} \subset X_\lambda \cup X_{\lambda'}$,(i) $x \sim_F y \sim_F z$ implies $x \sim_F z$ and (ii) either $x \succ_F y \succcurlyeq_F z$ or $x \succcurlyeq_F y \succ_F z$ implies $x \succ_F z$.

There are four cases for CFR.
(0) either $x, y, z \in X_\lambda$ or $x, y, z \in X_{\lambda'}$;
(1) $x \in X_\lambda, y \in X_\lambda, z \in X_{\lambda'}$;
(2) $x \in X_\lambda, y \in X_{\lambda'}, z \in X_{\lambda'}$;
(3) $x \in X_\lambda, y \in X_{\lambda'}, z \in X_\lambda$.

Case (0) is essentially equivalent to Full Rationality (FR) imposed on Arrovian rules[2]. An everyday example illustrates Case (1).

[2] A rule F satisfies FR if and only if $\succcurlyeq_F$ is complete and transitive for any $\succcurlyeq \in P(X)^n$.

Example 5 Let a non-welfare attribute be religion. Let x be a social state where we are Christian with a piece of bread per day, y be a social state where we are Christian with no bread per day, and z be a social state where we are not religious with bread as much as we like per day. Then it looks natural that $x \succcurlyeq_F y \succcurlyeq_F z$ implies $x \succcurlyeq_F z$. If we Christians like having one piece of bread better than no bread ($x \succcurlyeq_F y$) and if we like being a Christian with no bread better than being a rich with no religious faith ($y \succcurlyeq_F z$), then we like being a Christian with one piece of bread better than a rich with no religious faith ($x \succcurlyeq_F z$).

The formal meaning of Case (1) is as follows. Note that $\{x, z\}$ and $\{y, z\}$ have no difference in non-welfare attributes; $x \in X_\lambda$ and $z \in X_{\lambda'}$ whereas $y \in X_\lambda$ and $z \in X_{\lambda'}$. Thus if we have $x \prec_F z$ whereas $y \succcurlyeq_F z$, this implies that the welfare attributes of y are more highly praised in social preference than that of x. But this is a contradiction because $x \succcurlyeq_F y$ was made only by welfare attributes. In this case we can ignore non-welfare attributes since x and y have no difference in non-welfare attributes. Therefore $x \succcurlyeq_F z$ should be made. Case (2) can be justified as well.

A slight modification of the everyday example in Case (1) illustrates Case (3).

Example 6 Let x be a social state where we are Christian with a piece of bread per day, y be a social state where we are not religious with bread as much as we like per day, and z be a social state where we are Christian with no bread per day. If we like being a Christian with a piece of bread better than being a rich with no religious faith ($x \succcurlyeq_F y$) but we might as well discard the faith as starve to death ($y \succcurlyeq_F z$), then we Christian like having food better than no food ($x \succcurlyeq_F z$).

The formal meaning of Case (3) is explained as well as in Case (1). Note that $\{x, y\}$ and $\{y, z\}$ have no difference in non-welfare attributes; $x \in X_\lambda$ and $y \in X_{\lambda'}$ whereas $y \in X_{\lambda'}$ and $z \in X_\lambda$. Thus $x \succcurlyeq_F y \succcurlyeq_F z$ implies that the welfare attributes of x are not less praised in social preference than that of z. Since there exists no difference in non-welfare attributes between x and z, the social preference on $\{x, z\}$ should be made only by the welfare attributes so that we conclude $x \succcurlyeq_F z$.

Example 7 Let $X = \{x, y, z\}$. The attribute sets are $\{x, y\}$ and $\{z\}$. The table below shows that (1)-(3) are independent each other.

	(1)	(2)	(3)
$x \sim_F z \sim_F y \succ_F x$	yes	yes	no
$z \sim_F x \sim_F y \succ_F z$	yes	no	yes
$y \sim_F x \sim_F z \succ_F y$	no	yes	yes

As we noted before, transitivity of $\succcurlyeq_F$ does not always hold if three social states belong to different attribute sets. The example below shows this point.

Example 8 There exist three non-welfare attributes, religion, health and sex. Let x be a social state where we are Christian and smokers, and same-sex marriage is not legalized, y be a social state where we are Non-Christian and nonsmokers, and same-sex marriage is not legalized, and z be a social state where we are are Non-Christian and smokers, and same-sex marriage is legalized. The attribute sets are $\{x\}, \{y\}, \{z\}$. In this example $x \succcurlyeq_F y \succcurlyeq_F z$ does not imply $x \succcurlyeq_F z$. Note that the social decision for any two social states are made by two non-welfare attributes; $x \succcurlyeq_F y$ is made by religion and health whereas $y \succcurlyeq_F z$ is made by health and sex. Similarly $x \succcurlyeq_F z$ has to be made by sex and religion. But no information needed for this decision is contained in $x \succcurlyeq_F y$ and $y \succcurlyeq_F z$.

A rule F satisfies Binary Independence (BI) if for any $\succcurlyeq, \succcurlyeq' \in P(X)^n$ and any $x, y \in X$, if $\succcurlyeq_i \cap \{x,y\}^2 = \succcurlyeq'_i \cap \{x,y\}^2$ for all $i \in N$, then $\succcurlyeq_F \cap \{x,y\}^2 = \succcurlyeq'_F \cap \{x,y\}^2$. A rule F satisfies Binary Pareto (BP) if for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$, if $x \succ_i y$ for all $i \in N$, then $x \succ_F y$. A rule F satisfies Indifference Pareto (IP) if for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$, if $x \sim_i y$ for all $i \in N$ then $x \sim_F y$. A rule F is the Pareto extension rule if and only if for all $\succcurlyeq \in P(X)^n$ and all $x, y \in X$, $x \succcurlyeq_F y \Longleftrightarrow \neg (y \succ_i x \; \forall i \in N)$. A person i is decisive for (x, y) if for any $\succcurlyeq \in P(X)^n$, $x \succ_i y$ implies $x \succ_F y$. A person i is dictator on $Y \subset X$ if he is decisive for any pair in $Y \times Y$. A person i is dictator if he is dictator on X[3]. Binary Neutrality holds on $Y \subset X$ if for any $\succcurlyeq \in P(X)^n$ and any $x, y, z, w \in Y$, $\{i \in N : x \succcurlyeq_i y\} = \{i \in N : z \succcurlyeq_i w\}$ and $\{i \in N : x \preccurlyeq_i y\} = \{i \in N : z \preccurlyeq_i w\}$ imply $x \succcurlyeq_F y \Longleftrightarrow z \succcurlyeq_F w$. If Binary Neutrality holds on X, we say simply a rule F satisfies Binary Neutrality (BN).

For any $x \in X$, let $X(x)$ be the attribute set containing x. We say that a rule F uses non-welfare attributes if either (i) there exist some $\succcurlyeq \in P(X)^n$ and some $x, y \in X$ such that $X(x) \neq X(y)$, $x \sim_i y$ for all i and $x \nsim_F y$ or (ii) there exist some $\succcurlyeq \in P(X)^n$ and some $x, y, z, w \in X$ such that

(ii-a) $\{i \in N : x \succcurlyeq_i y\} = \{i \in N : z \succcurlyeq_i w\}$ and $\{i \in N : x \preccurlyeq_i y\} = \{i \in N : z \preccurlyeq_i w\}$;

(ii-b) $z \notin X(x) \cup X(y)$ or $w \notin X(x) \cup X(y)$; and

(ii-c) $x \succcurlyeq_F y \Longleftrightarrow z \succcurlyeq_F w$ does not hold.

Note that $x = z$ & $y = w$ never happens at (ii) because of (ii-b). Note also that if a rule uses non-welfare attributes it violates BN. We say that a rule F satisfies the Use of Non-Welfare Attributes (UNWA) if it uses non-welfare attributes. Note that if a rule satisfies UNWA then it violates BN, but not vice versa. The Borda rule violates BN but does not satisfy UNWA.

4 Results

Theorem 1 *(1) Suppose that there exists some X_λ with at least two elements. Then if a rule F satisfies CFR, BI and BP, there exists a person i who is decisive for any pair (x, y) except for all the pairs such that $\{x\} = X_\lambda$ and $\{y\} = X_{\lambda'}$.*

[3] We can say that i is dictator if he is decisive for all pairs in X.

(2) Suppose that any X_λ has at least two elements. Then if a rule F satisfies CFR, BI and BP, there exists dictator.

Proof. (1). Let X_λ be the set with x and y. Take $X_{\lambda'}(\lambda' \neq \lambda)$ and $z \in X_{\lambda'}$ arbitrarily. Thanks to CFR, $\succcurlyeq_F$ is complete and transitive on $\{x, y, z\}$ and hence Arrow's Theorem is applied. Thus there exists a dictator i on $\{x, y, z\}$. This further implies that i is dictator on X_λ and decisive for any pairs in $(X_\lambda \times X_{\lambda'}) \cup (X_{\lambda'} \times X_\lambda)$. We show i is dictator on $X_\lambda \cup X_{\lambda'}$ if $X_{\lambda'}$ contains at least two elements. Let $z, w \in X_{\lambda'}$ and $z \succ_i w$. We can let $z \succ_i x \succ_i w$ and $x \in X_\lambda$. Since i is decisive on $\{x, z\}$ and $\{x, w\}$, we have $z \succ_F x \succ_F w$. By CFR, we have $z \succ_F w$, a desired result. By noting that this holds for any $X_{\lambda'}$, this completes the proof of (1).

(2). (1) completes the proof. ■

Theorem 2 *If a rule satisfies CFR, BI, and UNWA, then it violates either FR or IP, and if there exist only two attribute sets, the rule satisfies FR and violates IP.*

Proof. The first part of the statement follows from a well known fact that any rule satisfying FR, BI and IP satisfies BN (Sen1970). Noting that FR is reduced to CFR for two attribute sets case, we establish the second part of the statement. ■

For rules satisfying all the axioms, Theorems 1 and 2 suggest that there are six cases that are logically possible. Table 1 lists the cases.

Table 1

	(1) of Th. 1	(2) of Th. 1
only two attribute sets IP is violated and FR is satisfied	Case 1	Case 2
three or more attribute sets FR is violated	Case 3	Case 4
three or more attribute sets IP is violated	Case 5	Case 6

Note that Case 5 is reduce to Case 3 since if FR is satisfied at Case 5, there exists dictator so that the conclusion of (1) of Theorem 1 does not hold. Thus FR is also violated at Case 5 which is reduced to Case 3. See Appendix for more detailed arguement on the remaining cases.

We show independence of the axioms. Each attribute set is indexed by X_τ $(\tau = 1, ..., t)$. For any $x \in X$, let $\tau(x) \in \{1, ..., t\}$ be such that $x \in X_{\tau(x)}$.

Example 9 Let $N(x, y, \succcurlyeq) = \#\{i \in N : x \succ_i y\}$. Let a rule F be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$x \succ_F y \Longleftrightarrow N(x, y, \succcurlyeq) > N(y, x, \succcurlyeq)$ or $[N(x, y, \succcurlyeq) = N(y, x, \succcurlyeq)$ and $\tau(x) > \tau(y)]$

$x \sim_F y \Longleftrightarrow N(x, y, \succcurlyeq) = N(y, x, \succcurlyeq)$ and $\tau(x) = \tau(y)$.

This rule has no dictator and satisfies all the axioms except for CFR.

Example 10 Let $\beta_i(x, \succcurlyeq) = \#\{y \in X : x \succcurlyeq_i y\}$ and $\beta(x, \succcurlyeq) = \sum_{i=1}^{n} \beta_i(x, \succcurlyeq)$. Let $k > 0$ be such that $n + k > kt$. Let a rule F be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$$x \succcurlyeq_F y \Longleftrightarrow \beta(x, \succcurlyeq) + k\tau(x) \geq \beta(y, \succcurlyeq) + k\tau(y).$$

This rule has no dictator and satisfies all the axioms except for BI. Note that BP is assured by the condition $n + k > kt$.

Example 11 Let a rule F be defined by: For any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,
$x \succ_F y \Longleftrightarrow [\tau(x) > \tau(y)]$ or $[\tau(x) = \tau(y) \& x \succ_1 y]$
$x \sim_F y \Longleftrightarrow \tau(x) = \tau(y) \& x \sim_1 y$
This rule has no dictator and satisfies all the axioms except for BP.

Example 12 Let a rule F be such that there exists some $i \in N$, called complete dictator, such that $x \succcurlyeq_F y \Longleftrightarrow x \succcurlyeq_i y$ for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$. This rule, called complete dictatorial rule, satisfies all the axioms except for UNWA.

5 Conclusion

6 Appendix

Cases 1, 2 and 6: Given $\succcurlyeq \in P(X)^n$, we define a lexicographic order $\succcurlyeq_L$ as follows.

For any $x, y \in X$, the asymmetric part of $\succcurlyeq_L$ is defined by

$$x \succ_L y \Longleftrightarrow \begin{cases} \exists k \in \{1, ..., n\} \text{ s.t. } x \sim_i y \ \forall i \leq k-1 \ \& x \succ_k y \\ \text{or} \\ x \sim_i y \ \forall i \ \& \ \tau(x) > \tau(y). \end{cases}$$

The symmetric part is defined by $x \sim_L y \Longleftrightarrow x \sim_i y \ \forall i \ \& \ \tau(x) = \tau(y)$.

Let a rule F be such that $x \succcurlyeq_F y \Longleftrightarrow x \succcurlyeq_L y$ for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$. Person 1 is dictator for F. This rule illustrates Cases 1, 2 and 6[4]. It is obvious that F satisfies FR, BI, BP and UNWA, and violates IP.

Case 3: Let a rule F be such that for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$$x \succcurlyeq_F y \Longleftrightarrow \begin{cases} x \succcurlyeq_1 y & \text{if } x, y \in X_\lambda \cup X_{\lambda'} \text{ with } \#X_\lambda \geq 2 \text{ or } \#X_{\lambda'} \geq 2 \\ \text{or} & \\ \neg\,(y \succ_i x \ \forall i \in N) & \text{otherwise.} \end{cases}$$

This rule illustrates Case 3. This rule satisfies CFR, BI, BP, UNWA and IP, and violates FR . Person 1 is decisive in (1) of Theorem 1. Let $X =$

[4] Note that the only possible attribute sets for Case 1 are $X - \{x\}$ and $\{x\}$ for any $x \in X$. Thus Person 1 is dictator for Case 1.

$\{x, y, z, w\}$ where the attribute sets are $\{x, y\}$, $\{z\}$ and $\{w\}$. According the rule, 1 is complete dictator[5] on $\{x, y, z\}$ and $\{x, y, w\}$ and the Pareto extension rule govern on $\{z, w\}$. Non-welfare attributes are used since $x \succ_1 y$ implies $x \succ_F y$ whereas $z \succ_1 w$ does not always imply $z \succ_F w$. FR is also violated since $z \succ_F x \succ_F w$ does not always imply $z \succ_F w$.

Case 4: Suppose that there exist at least three attribute sets. Let A, B, C be such that $A = \{x : \tau(x) = 1\}$, $B = \{x : \tau(x) = 2\}$, and $C = \{x : \tau(x) \geq 3\}$. A binary relation $\geq_T$ is defined by its asymmetric parts $>_T$and symmetric parts $=_T$as follows:

$$\begin{aligned} x >_T y &\iff [x \in A \& y \in B] \vee [x \in B \& y \in C] \vee [x \in C \& y \in A] \\ x =_T y &\iff [x, y \in A] \vee [x, y \in B] \vee [x, y \in C] \end{aligned}$$

Note that $\geq_T$ is complete but not transitive; $>_T$has cycles such that $x >_T y >_T z >_T x$ where $x \in A$, $y \in B$ and $z \in C$. Let F be such that for any $\succcurlyeq \in P(X)^n$ and any $x, y \in X$,

$$\begin{aligned} x \succ_F y &\iff x \succ_1 y \text{ or } [x \sim_1 y \text{ and } x >_T y], \\ x \sim_F y &\iff [x \sim_1 y \text{ and } x =_T y]. \end{aligned}$$

This rule illustrates Case 4. This is a dictatorial rule satisfying CFR, BI, BP and UNWA, and violates FR. Letting $x \in A$, $y \in B$, $z \in C$, $x \sim_1 y \sim_1 z$, we have $x \succ_F y \succ_F z \succ_F x$. This shows that F uses non-welfare attributes. It is easy to check that this rule satisfies CFR.

References

[1] Arrow KJ (1951,1963) Social Choice and Individual Values Wiley New York

[2] Gibbard A (1974) A Pareto-consistent libertarian claim, Journal of Economic Theory Vol.7 pp388-410

[3] Sen AK (1969) The impossibility of the Paretian liberal, Journal of Political Economy Vol.78 pp152-157

[4] Sen AK (1970) Collective Choice and Social Welfare Holden-Day SanFrancisco

[5] Sen AK (1976) Liberty, unanimity and rights Economica Vol.43. pp217-45

[6] Sen AK (1993) Internal consistency of choice Econometrica Vol.61. pp495-521

[5] We say that i is complete dictator on Y if for all $\succcurlyeq \in P(X)^n$ and all $x, y \in Y$, $x \succcurlyeq_i y \iff x \succcurlyeq_F y$.