LEARNING DYNAMICS

- RATIONAL EXPECTATIONS OFTEN DEFENDED AS PLAUSIBLE OUTCOME OF AGENTS' LEARNING THAT HAS CONVERGED.

⇒ EXPLICIT ANALYSIS OF LEARNING DYNAMICS?

- GENERAL "UNCERTAINTY PRINCIPLE": LEARNING DYNAMICS LOCALLY UNSTABLE NEAR A STEADY STATE WHEN AGENTS ARE UNCERTAIN ABOUT STABILITY, HENCE READY TO EXTRAPOLATE WIDE RANGE OF TRENDS OUT OF PAST DEVIATIONS FROM EQUILIBRIUM, INCLUDING NON CONVERGENT ONES, AND WHEN EXPECTATIONS MATTER MUCH FOR THE DYNAMICS.

  ONE MAY GET LOCAL STABILITY IF AGENTS ARE FAIRLY SURE ABOUT STABILITY (EXTRAPOLATE ONLY SMALL CONVERGENT TRENDS) AND/OR EXPECTATIONS DON'T MATTER MUCH.

- PRINCIPLE ARISES IN SMOOTH LEARNING, ERROR LEARNING, LEAST SQUARES LEARNING, BAYESIAN LEARNING

- LOCAL INSTABILITY DOES NOT NECESSARILY IMPLY GLOBAL EXPLOSIVE BEHAVIOR. ONE MAY GET GLOBAL CONVERGENCE OR EXISTENCE OF ATTRACTING MORE OR LESS COMPLEX "LEARNING EQUILIBRIA"

→ A. LOCAL (IN)STABILITY OF LEARNING DYNAMICS (SMOOTH CASE)

- TEMPORARY EQUILIBRIUM

(1) $T(x_{t-1}, x_t, x^e_{t+1}) = 0$, $T(\bar{x}, \bar{x}, \bar{x}) = 0$

$\quad b_1 \quad b_0 \quad a \neq 0$

- EXPECTATION FUNCTION (AVERAGE)

(2) $x^e_{t+1} = \psi(x_t, \ldots, x_{t-L})$, $\bar{x} = \psi(\bar{x}, \ldots, \bar{x})$

$\quad c_0, \ldots, c_L$

- LEARNING DYNAMICS: (1) + (2) ⇒

(3) $T(x_{t-1}, x_t, \psi(x_t, \ldots, x_{t-L})) = 0$

$\quad b_0 \quad a c_0$

$b_0 + a c_0 \neq 0 \Rightarrow$ (IMPLICIT FUNCTION THEOREM)

(4) $x_t = W_{loc}(x_{t-1}, \ldots, x_{t-L})$

- LOCAL STABILITY OF $\bar{x} = W_{loc}(\bar{x}, \ldots, \bar{x}) \Leftrightarrow$

ALL ROOTS OF CHARACTERISTIC POLYNOMIAL

$$Q_W(z) \equiv b_1 z^{L-1} + b_0 z^L + a \sum_0^L c_j z^{L-j} = 0$$

HAVE MODULUS < 1 (LINEARIZED DYNAMICS)

- DECOMPOSITION

$$Q_W(z) \equiv z^{L-1}(b_1+b_0 z+a z^2)-a\left(z^{L+1}-\sum_0^L c_j z^{L-j}\right)=0$$

$$\equiv z^{L-1} Q_F(z) - a Q_\psi(z)=0$$

- INTERPRETATION ?

⇒ $Q_F(z) \equiv b_1+b_0 z+a z^2=0$ IS CHARACTERISTIC POLYNOMIAL OF LOCAL PERFECT FORESIGHT DYNAMICS $T(x_{t-1}, x_t, x_{t+1})=0$. LOCAL PERFECT FORESIGHT ROOTS λ_1, λ_2.

⇒ $Q_\psi(z) \equiv z^{L+1}-\sum_0^L c_j z^{L-j}=0$ IS CHARACTERISTIC POLYNOMIAL OF EXPECTATIONS FUNCTION. LOCAL EIGENVALUES OF AVERAGE FORECASTING RULE: $\mu_1, \dots, \mu_{L+1}$

- <u>QUALITATIVE CONCLUSION:</u>

LOCAL STABILITY OF LEARNING DYNAMICS DEPENDS ONLY ON HOW LOCAL EIGENVALUES $\{\mu_j\}$ OF AGENTS' AVERAGE FORECASTING RULE INTERACT WITH THE LOCAL PERFECT FORESIGHT ROOTS λ_1, λ_2.

- INTERPRETATION OF LOCAL EIGENVALUES $\{\mu_j\}$?

$$Q_\psi(z) \equiv z^{L+1} - \sum_0^L c_j z^{L-j} = 0$$

$\Leftrightarrow$ LOCAL LINEAR APPROXIMATION OF ψ NEAR $\bar{x}$
$\sim$ "LINEAR FILTER" ON TRENDS AND FREQUENCIES APPEARING IN PAST DEVIATIONS $\Delta x_{t-j} = x_{t-j} - \bar{x}$.

$$\left(\Delta x^e_{t+1} - \sum_0^L c_j \Delta x_{t-j} = 0 \right)$$

- IF ψ EXTRAPOLATES LOCALLY CONSTANT SEQUENCES $(\Delta x^e_{t+1} = x - \bar{x}$ WHEN $\Delta x_{t-j} = x - \bar{x}\ \forall j) \Leftrightarrow Q_\psi(1) = 0$
- IF ψ EXTRAPOLATES LOCALLY CYCLES OF PERIOD 2 $\Leftrightarrow Q_\psi(\pm 1) = 0$ (EX: NAIVE $x^e_{t+1} = x_{t-1}$ OR $x_2 = \psi(x_1, x_2, x_1, x_2, \dots)$)
- IF ψ EXTRAPOLATES LOCALLY TREND r AND CYCLICAL COMPONENT θ FROM PAST DEVIATIONS $\Leftrightarrow Q_\psi(re^{i\theta}) = 0$

$\Longrightarrow$ LOCAL EIGENVALUES $\{\mu_j\}$ = SET OF TRENDS AND FREQUENCIES AGENTS ARE ON AVERAGE ABLE TO DETECT AND WILLING TO EXTRAPOLATE FROM PAST DEVIATIONS $\Delta x_{t-j} = x_{t-j} - \bar{x}$

$\Rightarrow$ LOCAL INSTABILITY OF LEARNING DYNAMICS IF AGENTS ARE ON AVERAGE UNCERTAIN ABOUT STABILITY, HENCE WILLING TO EXTRAPOLATE A LARGE SET OF TRENDS AND FREQUENCIES FROM PAST DEVIATIONS $\Delta x_{t-j} = x_{t-j} - \bar{x}$, INCLUDING NON-CONVERGENT ONES ($|\mu_j| \geq 1$), AND IF EXPECTATIONS MATTER SIGNIFICANTLY ($|a|$ LARGE)

<u>EX</u>: $x^e_{t+1} = x_{t-1}$ (NAIVE, $L=1$, $\mu_j = \pm 1$) AND $|a|$ LARGE $\Rightarrow Q_W(z) \equiv b_1 + b_0 z + a \Rightarrow$ ROOT OF Q_W HAS LARGE MODULUS

<u>PROPOSITION 1</u>: IF $\mu_1 < 0 < \mu_2$ AND IF $[\mu_1, \mu_2]$ CONTAINS IN ITS INTERIOR ALL PERFECT FORESIGHT ROOTS λ_1, λ_2 THAT ARE REAL, THEN Q_W HAS REAL ROOT r SUCH THAT EITHER $r < \mu_1$ OR $r > \mu_2$.

$\Rightarrow$ INSTABILITY IF $\mu_1 \leq -1$ AND $\mu_2 \geq 1$

PROOF: $a\,Q_F(\mu_i) > 0,\ i = 1,2 \Rightarrow Q_F(\mu_1), Q_F(\mu_2)$ SAME SIGN

SIGN $(-1)^L Q_W(\mu_1) = -$ SIGN $Q_F(\mu_1)$ AND

SIGN $Q_W(\mu_2) =$ SIGN $Q_F(\mu_2)$

$\Rightarrow (-1)^L (b_0 + a c_0) Q_W(\mu_1)$ AND $(b_0 + a c_0) Q_W(\mu_2)$ HAVE OPPOSITE SIGN $\Rightarrow Q_W(z)$ HAS ROOT r WITH EITHER $r > \mu_2$ OR $r < \mu_1$. Q.E.D.

$\Rightarrow$ LOCAL STABILITY OF LEARNING DYNAMICS IF AGENTS ARE ON AVERAGE SURE ABOUT (ABLE TO CONTROL COLLECTIVELY) STABILITY, HENCE WILLING TO EXTRAPOLATE ONLY (SMALL) CONVERGENT TRENDS AND IF EXPECTATIONS DON'T MATTER MUCH ($|a|$ SMALL)

EX: IF ALL $|\mu_j|$, HENCE ALL $|c_j|$, SMALL

IF $|a|$ SMALL ($\Rightarrow Q_F(z) \simeq b_1 + b_0 z \Rightarrow |\lambda_1| \to +\infty$, $|\lambda_2| \to |b_1/b_0| \Rightarrow$ SADDLE IN PERFECT FORESIGHT DYNAMICS WHEN $|b_0| > |b_1|$

THEN $Q_W(z) \simeq z^{L-1}(b_1 + b_0 z) \Rightarrow$ LOCAL STABILITY IF $|b_0| > |b_1|$

PROPOSITION 2: LET $b_0 + a c_0 \neq 0$, $\alpha = \mathrm{MAX} \left| \sum_0^L c_j z^{L-j} \right|$ FOR $|z| = 1$.

1. LOCAL STABILITY OF LEARNING DYNAMICS IF $|b_0| > |b_1| + |a|\alpha$

2. IF ψ EXTRAPOLATES CONSTANT SEQUENCES ($Q_\psi(1) = 0$) AND $c_j \geqq 0 \ \forall j \Rightarrow \alpha = 1$ AND LOCAL STABILITY IF $|b_0| > |b_1| + |a|$

PROOF: 1. IF $Q_W(\bar{z}) = 0$ FOR $|\bar{z}| \geqq 1$,

$$|b_0| = \left| b_1 \bar{z}^{-1} + a \sum_0^L c_j \bar{z}^{-j} \right| \leqq |b_1| + |a|\alpha$$

2. IF $c_j \geqq 0 \ \forall j$ AND $\sum_j c_j = 1$, $\max_{|z|=1} \left| \sum_0^L c_j z^{L-j} \right| = 1$.

- QUALITATIVE CONCLUSION: "UNCERTAINTY PRINCIPLE"

- IF AGENTS ARE ON AVERAGE UNCERTAIN ABOUT STABILITY $\Rightarrow$ WIDELY SPREAD SET OF LOCAL EINGEN VALUES OF AVERAGE EXPECTATION FUNCTION $\Longrightarrow$ LOCAL INSTABILITY OF LEARNING DYNAMICS IF EXPECTATIONS MATTER MUCH ($|a|$ LARGE)
- IF AGENTS ARE SURE ABOUT STABILITY $\Rightarrow$ EXTRAPOLATE ONLY SMALL CONVERGENT TRENDS (α SMALL) $\Rightarrow$ LOCAL STABILITY OF LEARNING DYNAMICS MAY OCCUR IF EXPECTATIONS DON'T MATTER MUCH ($|a|$ SMALL) (IF $|b_0| > |b_1|$)

- OPPOSITE (COUNTERFACTUAL?) CONCLUSIONS FOR PERFECT FORESIGHT DYNAMICS

- $|a|$ LARGE $\Rightarrow Q_F(z) \simeq a z^2 \Rightarrow |\lambda_1|, |\lambda_2|$ SMALL
- $|a|$ SMALL $\Rightarrow Q_F(z) \simeq b_1 + b_0 z \Rightarrow$ $|\lambda_1| \to +\infty$, $|\lambda_2| \to |b_1/b_0|$

UNDER PERFECT FORESIGHT, MARKETS FOR WHICH EXPECTATIONS MATTER MOST ($|a|$ LARGE) SHOULD BE MORE STABLE

→ B. ERROR LEARNING

- CASE WHERE FORECASTS ARE REVISED FROM PAST FORECASTING ERRORS (~ INFINITE MEMORY)

– <u>EX</u>: $x^e_{t+1} = c\,x_{t-1} + (1-c)\,x^e_{t-1} = x^e_{t-1} + c\,(x_{t-1} - x^e_{t-1})$

WITH $0 < c < 1$

$\Longrightarrow$ SAME CONCLUSIONS ("UNCERTAINTY PRINCIPLE")

- TEMPORARY EQUILIBRIUM MAP

$$(1)\quad T(x_{t-1}, x_t, x^e_{t+1}) = 0 \quad , \quad T(\underset{b_0}{\bar{x}}, \underset{b_1}{\bar{x}}, \underset{a \neq 0}{\bar{x}}) = 0$$

- EXPECTATION FUNCTION

$$(2)\quad x^e_{t+1} = \psi^*\begin{pmatrix} x_t, & x_{t-1}, & \dots, & x_{t-L} \\ & x^e_t, & \dots, & x^e_{t-L+1} \end{pmatrix} \begin{matrix} (c^*_j) \\ (\gamma^*_j) \end{matrix}$$

- LEARNING DYNAMICS : (1)+(2) DEFINES IMPLICITLY STATE $y_t = \begin{pmatrix} x_t \\ x^e_{t+1} \end{pmatrix}$ AS FUNCTION OF PAST STATES $y_{t-j} = \begin{pmatrix} x_{t-j} \\ x_{t-j+1} \end{pmatrix}$. IF $b_0 + a\,c^*_0 \neq 0$

$$(3)\quad \begin{pmatrix} x_t \\ x^e_{t+1} \end{pmatrix} = W_{loc}\begin{pmatrix} x_{t-1}, \dots, x_{t-L} \\ x^e_t, \dots, x_{t-L+1} \end{pmatrix}$$

(IMPLICIT FUNCTION THEOREM)

- CHARACTERISTIC POLYNOMIAL (DEGREE 2L)

$$Q_W(z) \equiv \left(b_1 z^{L-1} + b_0 z^L\right)\left(z^L - \sum_1^L \gamma_j^* z^{L-j}\right) - a z \sum_0^L c_j^* z^{L-j} = 0$$

- DECOMPOSITION

– ERROR LEARNING CHARACTERISTIC POLYNOMIAL

$$Q_e(z) \equiv z^L - \sum_1^L \gamma_j^* z^{L-j} = 0$$

IS CHARACTERISTIC POLYNOMIAL OF

$$x^e_{t+1} = \psi^*\left(\begin{matrix} \bar{x}, \bar{x}, \dots, \bar{x} \\ x^e_t, \dots, x^e_{t-L+1} \end{matrix} \right)$$

MODULUS OF ROOTS OF Q_e MEASURE WHETHER (<1) AND HOW FAST AGENTS LEARN FROM PAST ERRORS

$\Rightarrow$ BY $\pm a z^{L+1} Q_e(z)$ TO $Q_W(z)$:

$$\underline{Q_W(z) \equiv z^{L-1} Q_F(z)\, Q_e(z) - a z^L Q_\psi(z) = 0}$$

WHERE $Q_F(z)$ IS PERFECT FORESIGHT CH. POL. (λ_1, λ_2)

AND

$$-Q_\psi(z) \equiv z^{L+1} - \sum_0^L c_j z^{L-j} = 0, \quad c_j = c_j^* + \gamma_{j+1}^* \text{ is}$$

CHARACTERISTIC POLYNOMIAL OF EXPECTATION FUNCTION

$$x^e_{t+1} = \psi\left(\begin{matrix} x_t, x_{t-1}, \dots, x_{t-L} \\ x_t, \dots, x_{t-L+1} \end{matrix} \right)$$

OBTAINED WHEN NO FORECASTING ERRORS,

$$x^e_{t-j+1} = x_{t-j+1}, \quad j = 1, \dots, L$$

$\Rightarrow$ LOCAL EIGENVALUES OF ψ = SAME INTERPRETATION AS BEFORE

• QUALITATIVE CONCLUSION:

LOCAL STABILITY OF LEARNING DYNAMICS DEPENDS ONLY ON HOW THE LOCAL EIGENVALUES $\{\mu_j\}$ (TRENDS AND FREQUENCIES AGENTS EXTRAPOLATE FROM PAST DEVIATIONS $x_{t-j} - \bar{x}$ "CORRECTLY", IE WHEN THERE ARE NO FORECASTING ERRORS, $x^e_{t-j+1} = x_{t-j+1}$) INTERACT WITH PERFECT FORESIGHT ROOTS (λ_1, λ_2) AND WITH HOW FAST AGENTS LEARN FROM PAST ERRORS (WITH ROOTS OF ERROR LEARNING POLYNOMIAL $Q_e(z)$)

INSTABILITY

PROPOSITION 1': IF $\mu_1 < 0 < \mu_2$ AND IF $[\mu_1, \mu_2]$ CONTAINS IN ITS INTERIOR ALL REAL PERFECT FORESIGHT ROOTS λ_1, λ_2 AS WELL AS ALL REAL ROOTS OF ERROR LEARNING POLYNOMIAL $Q_e(z)$, THEN Q_W HAS REAL ROOT r SUCH THAT EITHER $r < \mu_1$ OR $r > \mu_2$.

$\Rightarrow$ INSTABILITY IF $\mu_1 \leqq -1$ AND $\mu_2 \geqq 1$.

WILL OCCUR IF AGENTS UNCERTAIN ABOUT STABILITY ($[\mu_1, \mu_2]$ LARGE), EXPECTATIONS MATTER MUCH ($|a|$ LARGE, λ_1, λ_2 SMALL) AND AGENTS LEARN FAST (γ^*_j SMALL)

INSTABILITY

PROPOSITION 2': LET ALL ROOTS OF ERROR LEARNING POLYNOMIAL $Q_e(z)$ HAVE MODULUS<1, AND

$$\alpha^* = \max_{|z|=1} \left|\sum_0^L c_j^* z^{L-j}\right| / |Q_e(z)|$$

$\Rightarrow$ LOCAL STABILITY IF $|b_0| > |b_1| + |a|\alpha^*$

WILL OCCUR IF AGENTS LEARN FAST (ALL γ_j^* SMALL $\Leftrightarrow$ MODULUS OF ROOTS OF $Q_e(z)$ SMALL), ARE FAIRLY SURE ABOUT STABILITY (EXTRAPOLATE ONLY SMALL CONVERGENT TRENDS $\Leftrightarrow$ α^* SMALL) AND EXPECTATIONS DON'T MATTER MUCH ($|a|$ SMALL), WHENEVER $|b_0| > |b_1|$.

$\Rightarrow$ SAME QUALITATIVE "UNCERTAINTY PRINCIPLE"

EX: NAIVE ERRORLEARNING

$$x^e_{t+1} = c\, x_{t-1} + (1-c)\, x^e_{t-1} = x^e_{t-1} + c\,(x_{t-1} - x^e_{t-1})$$

$c_0^* = 0,\ c_1^* = c,\ \gamma_1^* = 0,\ \gamma_2^* = (1-c)$

$\psi \equiv x_{t-1} \Rightarrow \mu_1 = -1,\ \mu_2 = +1$ AND $\alpha^* = 1$

$Q_W(z) \equiv z\,[\,b_1 + b_0 z^3 - z\,(b_0(1-c) + ac)\,] = 0$

$|a|$ LARGE $\Rightarrow$ UNSTABLE

$|a|$ SMALL AND $|1-c|$ SMALL $\Rightarrow$ STABLE IF $|b_0| > |b_1|$

→ G. LEAST SQUARES LEARNING (DISCONTINUOUS CASE)

• TEMPORARY EQUILIBRIUM MAP

(1) $b_1 x_{t-1} + b_0 x_t + a x^e_{t+1} = 0$

$\bar{x} = 0$, x_t = DEVIATION

• EXPECTATION FUNCTION : FOR $x_{t-j} \neq 0$

(2) $x^e_{t+1} = \beta^2_t x_{t-1}$ WHERE β_t IS SOME AVERAGE OF PAST GROWTH RATES x_{t-j+1}/x_{t-j} FOR $j = 2, \ldots$

• LEARNING DYNAMICS : (1)+(2) ⇒ FOR $x \neq 0$

(3) $x_t / x_{t-1} = -\dfrac{b_1 + a\beta^2_t}{b_0} \equiv \Omega(\beta_t)$

- "PROJECTION FACILITY" $[\mu_1, \mu_2]$. IF β_t^* IS UNCONSTRAINED AVERAGE OF PAST GROWTH RATES x_{t-j+1}/x_{t-j}, ACTUAL β_t IS "PROJECTION" OF β_t^* ON $[\mu_1, \mu_2]$ ($\beta_t = \beta_t^*$ IF $\in [\mu_1, \mu_2]$, $= \mu_1$ IF $\beta_t^* \leqq \mu_1$, $= \mu_2$ IF $\beta_t^* \geqq \mu_2$)

- EXAMPLES

- SIMPLE "LEARNING": $\beta_t^* = x_{t-1}/x_{t-2}$
- BELIEF $x_\tau / x_{\tau-1} = \beta + \eta_\tau$ (NOISE)

$$\Rightarrow \beta_t^* = \frac{1}{L-1} \sum_2^L x_{t-j+1}/x_{t-j} \quad \text{(OLS)}$$

- BELIEF $x_\tau = \beta x_{\tau-1} + \eta_\tau$ (NOISE)

$$\Rightarrow \beta_t^* = \sum_2^L x_{t-j+1} x_{t-j} \Big/ \sum_2^L x_{t-j}^2 \quad \text{(OLS)}$$

OR RECURSIVE FORMULATION (WHEN $L \to t+L$)

$$\beta_t^* = m_{t-1} \beta_{t-1}^* + (1 - m_{t-1})(x_{t-1}/x_{t-2})$$

WITH WEIGHT $0 < m_{t-1} < 1$ FUNCTION OF PAST GROWTH RATES x_{t-j+1}/x_{t-j}, $j = 2, \dots, t+L$

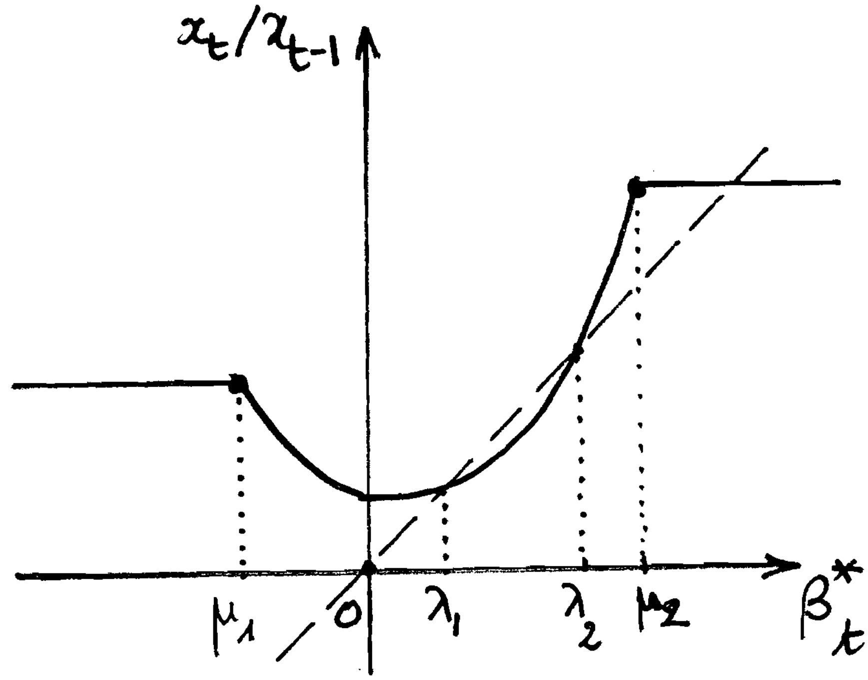
- EXPECTATIONS FUNCTION

$$(x_{t-1}, \dots, x_{t-L}) \to \beta_t^* \to \beta_t \to x_{t+1}^e = \beta_t^2 x_{t-1}$$

EXTRAPOLATES CORRECTLY CONTINUUM OF TRENDS $x_{t-j+1}/x_{t-j} = r$ IN $[\mu_1, \mu_2]$ ($\Rightarrow$ DISCONTINUOUS)

$\Rightarrow$ LOCAL INSTABILITY: IF $\mu_1 \leqq 1$, $\mu_2 \geqq 1$ AND IF $[\mu_1, \mu_2]$ CONTAINS IN ITS INTERIOR ALL PERFECT FORESIGHT ROOTS λ_1, λ_2 THAT ARE REAL, LEARNING DYNAMICS IS LOCALLY DIVERGENT FOR OPEN CONE OF INITIAL CONDITIONS

CASE $\lambda_1 + \lambda_2 = -b_0/a > 0$



$$x_t/x_{t-1} = \Omega(\beta_t) \quad \text{WITH} \quad \Omega(\beta) = -\frac{b_1 + a\beta^2}{b_0}$$

$$\beta_t = \beta^*_t \text{ IF } \beta^*_t \in [\mu_1, \mu_2]$$
$$= \mu_1 \text{ IF } \beta^*_t \leqq \mu_1, \; = \mu_2 \text{ IF } \beta^*_t \geqq \mu_2$$

IF $x_{-1}/x_{-2} \geqq \mu^*, \ldots, x_{-L+1}/x_{-L} \geqq \mu^*$ WITH

$\lambda_2 < \mu^* < \mu_2 \Rightarrow x_t/x_{t-1} \geqq \Omega(\mu^*) > \mu^* \;\; \forall t \geqq 0$

$\Rightarrow$ LOCAL DIVERGENCE IF $\Omega(\mu^*) > 1$

- ABOVE INSTABILITY EASY TO OBTAIN FOR LARGE OPEN CONE OF INITIAL DEVIATIONS WHEN EXPECTATIONS MATTER MUCH ($|a|$ LARGE) EVEN IF $-1 < \mu_1 < 0 < \mu_2 < 1$ ($|\Omega(\beta)|$ LARGE FOR $\beta \neq 0$ WHEN $|a|$ LARGE)

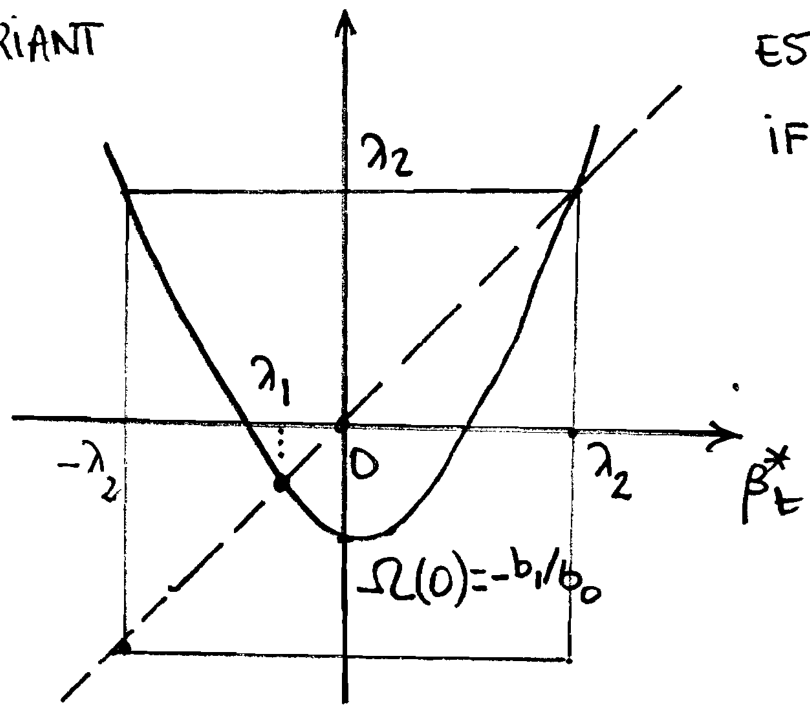
- ABOVE INSTABILITY MAY CO-EXIST WITH LOCAL STABILITY FOR ANOTHER OPEN CONE OF INITIAL DEVIATIONS EVEN WHEN PROJECTION FACILITY $[\mu_1, \mu_2]$ IS LARGE, IF EXPECTATIONS DON'T MATTER MUCH ($|a|$ SMALL) AND $|b_1/b_0| < 1$

– WHEN INITIAL GROWTH RATES LIE CLOSE TO λ_1 REAL WITH $|\Omega'(\lambda_1)| < 1$ AND $|\lambda_1| < 1$

CASE $\lambda_1 + \lambda_2 = -b_0/a > 0$

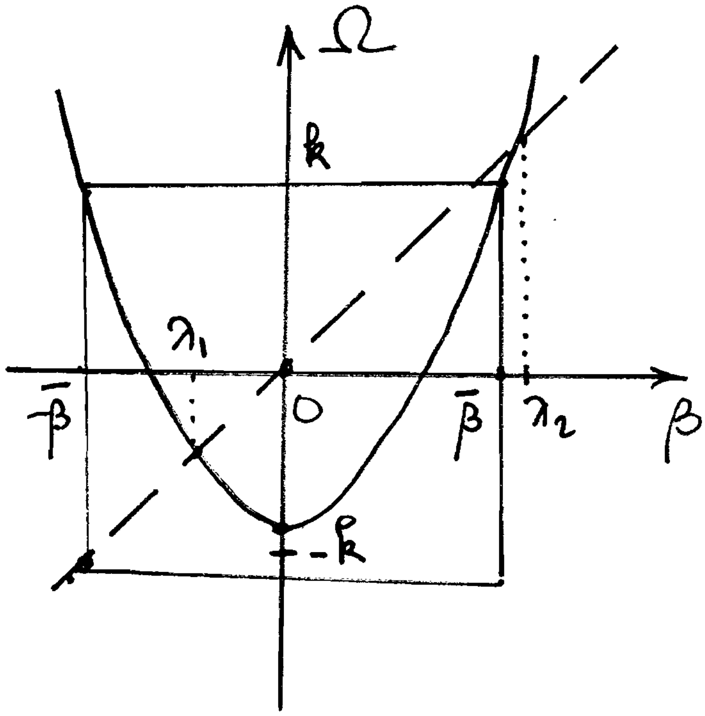
$[-\lambda_2, \lambda_2]$ INVARIANT BY Ω

ESTIMATE $\beta_t \to \lambda_1$ IF FINITE MEMORY L



– WHEN INITIAL GROWTH RATES LIE IN INTERVAL $B = [-\bar\beta, \bar\beta]$ THAT IS INVARIANT BY Ω AND $|\Omega(\beta)| \leq k < 1$ IN $B \Rightarrow \beta_t$, $x_t / x_{t-1} = \Omega(\beta_t)$ LIE IN B FOR ALL $t \geq 0 \Rightarrow x_t \to 0$ (BUT $\beta_t \not\to \lambda_1$)

CASE $\lambda_1 + \lambda_2 = -b_0/a > 0$



- FOR GIVEN PROJECTION FACILITY $[\mu_1, \mu_2]$, ONE WILL GET LOCAL STABILITY FOR ALL INITIAL DEVIATIONS $x_{-1} \neq 0, \dots, x_{-L} \neq 0$ ($x_t \to 0$ BUT β_t MAY NOT $\to \lambda_1$) IF $|a|$ SMALL ENOUGH, I.E. WHEN $|b_0| > |b_1| + |a|\alpha$ WITH $\alpha = \text{MAX}\{\mu_1^2, \mu_2^2\}$

PROOF: $|\Omega(\beta)| \leq (|b_1| + |a|\alpha / |b_0|) < 1$ FOR ALL β in $[\mu_1, \mu_2]$

$\Rightarrow$ QUALITATIVE CONCLUSION: "UNCERTAITY PRINCIPLE"

- IF AGENTS ARE UNCERTAIN ABOUT STABILITY ($[\mu_1, \mu_2]$ LARGE) $\Rightarrow$ LOCAL INSTABILITY FOR LARGE CONE OF INITIAL DEVIATIONS IF EXPECTATIONS MATTER MUCH ($|a|$ LARGE)
- LOCAL INSTABILITY MAY COEXIST WITH LOCAL STABILITY FOR ANOTHER CONE OF INITIAL DEVIATIONS. PROBABILITY OF STABILITY LARGER IF AGENTS FAIRLY SURE ABOUT STABILITY ($[\mu_1, \mu_2]$ CLOSER TO STABLE ROOT $|\lambda_1|<1$ WITH $|\Omega'(\lambda_1)|<1$) AND/OR EXPECTATIONS DON'T MATTER MUCH ($|a|$ SMALL ENOUGH, $|b_0|>|b_1|$)

D. BAYESIAN LEARNING

- TEMPORARY EQUILIBRIUM MAP

(1) $b_1 x_{t-1} + b_0 x_t + a x^e_{t+1} = 0$

$\bar{x} = 0$ STEADY STATE, x_t = DEVIATIONS

- EXPECTATION FUNCTION ($x_{t-j} \neq 0$)

– BELIEF $x_\tau = \beta x_{\tau-1} + \eta_\tau$, η_τ IID $\sim \mathcal{N}(0, \sigma^2_\eta)$

– PRIOR AT BEGINNING PERIOD t: $\beta \sim \mathcal{N}(\beta_t, \sigma^2_t)$, $\mathrm{cov}(\beta, \eta_t) = 0$

$\Rightarrow x^e_{t+1} = E_t[x_{t+1}] = E_t[\beta(\beta x_{t-1} + \eta_t) + \eta_{t+1}]$

(2) $x^e_{t+1} = E_t[\beta^2 x_{t-1}] = (\beta^2_t + \sigma^2_t) x_{t-1}$

– BAYESIAN UPDATING ($\sim$ OLS)

(2') $\beta_{t+1} = m_t \beta_t + (1 - m_t) x_t / x_{t-1}$, $\sigma^2_{t+1} = m_t \sigma^2_t$

WHERE $0 < m_t < 1$ FUNCTION OF PAST GROWTH RATES $x_{t-1}/x_{t-2}, \ldots, x_0/x_{-1}$ AND INITIAL PRIOR (β_0, σ_0)

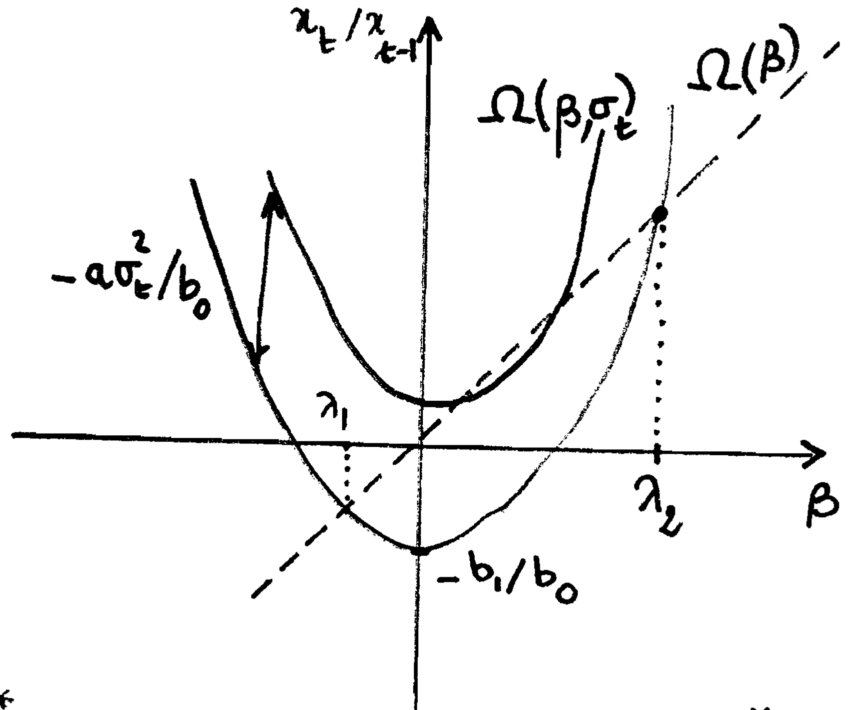
- LEARNING DYNAMICS: (1)+(2) $\Rightarrow$

(3) $$x_t / x_{t-1} = - \frac{b_1 + a(\beta^2_t + \sigma^2_t)}{b_0} \equiv \Omega(\beta_t, \sigma_t)$$

PLUS UPDATING (2'), GIVEN $x_{-1} \neq 0, (\beta_0, \sigma_0)$

LOCAL INSTABILITY

CASE $\lambda_1+\lambda_2 = -b_0/a > 0$



LET μ^* LARGER THAN λ_1, λ_2 WITH $\Omega(\mu^*)>1$

1. IF $\beta_0 \geqq \mu^* \Rightarrow x_0/x_{-1} \geqq \Omega(\beta_0) \geqq \beta_0 \Rightarrow \beta_1 \geqq \beta_0$

$\Rightarrow \forall t \geqq 0 \quad \beta_t \geqq \beta_{t-1}, \; x_t/x_{t-1} \geqq \Omega(\beta_t) \geqq \Omega(\beta_0) \geqq \Omega(\mu^*) > 1$

$\Rightarrow x_t$ DIVERGES

2. GIVEN β_0, IF σ_0 IS LARGE ENOUGH, β_1 IS CLOSE TO

$\Omega(\beta_0, \sigma_0) \geqq \mu^*$ (m_0 IS SMALL) $\Rightarrow$ BACK TO CASE 1

$\Rightarrow x_t$ DIVERGES

$\Rightarrow$ LOCAL INSTABILITY WHEN AGENTS BELIEVE IN INSTABILITY (β_0 LARGE) OR ARE UNCERTAIN ABOUT STABILITY (β_0 SMALL BUT σ_0 LARGE). EASY TO OBTAIN WHEN EXPECTATIONS MATTER MUCH ($|a|$ LARGE $\Rightarrow$ $|\Omega(\beta)|$ LARGE FOR $\beta \neq 0$)

LOCAL STABILITY

CONVERGENCE ($x_t \to 0$, $\beta_t \to \bar{\beta}$, MAY BE $\bar{\beta} \neq \lambda_1, \lambda_2$, $\sigma_t \downarrow \bar{\sigma}$, MAY BE $\bar{\sigma} > 0$) WHEN $x_{-1} \neq 0$ SMALL AND INITIAL PRIOR (β_0, σ_0) IS SUCH THAT

$$|b_0| > |b_1| + |a| (\beta_0^2 + \sigma_0^2)$$

WHEN $|b_0| > |b_1|$, SUFFICIENT CONDITION MET WHEN EXPECTATIONS DON'T MATTER TOO MUCH ($|a|$ SMALL, $\Rightarrow$ $|\lambda_2|$ LARGE, $|\lambda_1|$ CLOSE TO $|b_1/b_0| < 1$), WHEN AGENTS BELIEVE INITIALLY IN STABILITY ($|\beta_0|$ SMALL) AND HOLD THAT INITIAL BELIEF WITH ENOUGH CONFIDENCE (σ_0 SMALL)

$\Rightarrow$ QUALITATIVE CONCLUSION: "UNCERTAINTY PRINCIPLE"

E. GLOBAL STABILITY (LEAST SQUARES)

- TEMPORARY EQUILIBRIUM MAP

(1) $$x_t = F(x^e_{t+1}), \quad 0 = F(0)$$

($\bar{x}$ = STEADY STATE, x_t = DEVIATION)

(~ BEFORE WITH $a = F'(0)$, $b_0 = -1$, $b_1 = 0 \Rightarrow \lambda_1 = 0$, $\lambda_2 = 1/a$)

- EXPECTATION FUNCTION: FOR $x_{t-j} \neq 0$

(2) $x^e_{t+1} = \beta^{*2}_t x_{t-1}$ WHERE β^*_t IS (UNCONSTRAINED)

OLS ESTIMATE OF BELIEF $x_\tau = \beta x_{\tau-1} + \eta_\tau$ (NOISE), OR IN RECURSIVE FORMULATION

(2') $$\beta^*_t = m_{t-1}\beta^*_{t-1} + (1 - m_{t-1})\, x_{t-1}/x_{t-2}$$

WITH WEIGHTS $0 < m_{t-1} < 1$ FUNCTIONS OF PAST GROWTH RATES x_{t-j+1}/x_{t-j}, $j = 2, \dots, t+L$

- LEARNING DYNAMICS: (1)+(2) $\Rightarrow$

(3) $$x_t = F(\beta^{*2}_t x_{t-1})$$

PLUS UPDATING (2')

- PREVIOUS LOCAL ANALYSIS (FOR LINEAR $F(x) = ax$)

- LOCAL INSTABILITY FOR OPEN CONE OF INITIAL DEVIATIONS. IF $a > 0$ AND INITIAL β_0 (AVERAGE OF $x_{-1}/x_{-2}, \dots, x_{-L+1}/x_{-L}$) IS SUCH THAT $\beta_0 > \lambda_2 = 1/a$ AND $F(\beta_0^2 x)/x = a\beta_0^2 > 1$, ONE GETS DIVERGENCE $x_t/x_{t-1} \geq a\beta_0^2 > 1 \quad \forall t \geq 0$

- CO-EXISTS WITH LOCAL STABILITY FOR ANOTHER OPEN CONE OF INITIAL DEVIATIONS. IF $a > 0$ AND INITIAL β_0 (AVERAGE OF $x_{-1}/x_{-2}, \dots, x_{-L+1}/x_{-L}$) IS CLOSE TO $\lambda_1 = 0$ (HERE $\Omega'(0) = 0$) $\Rightarrow x_t \to 0$

- END OF STORY IF $F(x)$ IS GLOBALLY LINEAR. BUT IF $F(x)$ NONLINEAR WITH NICE GLOBAL "CONTRACTING" PROPERTIES, AND PROPERLY "BOUNDED", ONE MAY GET "GLOBAL STABILITY IN SPITE OF LOCAL INSTABILITY": LOCALLY DIVERGENT TRAJECTORIES MAY BE FORCED TO COME BACK AND OWING TO NON-LINEARITIES, BE TRAPPED BACK INTO STABILITY CONE.

PROPOSITION (CHATTERJI AND CHATTOPADHYAY, 2000)

ASSUME THAT F IS GLOBALLY CONTRACTING, $|F(x)| \leqq a|x|$ FOR $0<a<1$, $\forall x$ ($\Rightarrow$ EXPECTATIONS DO NOT MATTER TOO STRONGLY), THAT $F(x)$ IS BOUNDED BELOW, $F(x) \geqq -K$ FOR $K>0, \forall x$, AND BOUNDED ABOVE FOR POSITIVE x, $F(x) \leqq Q$ FOR $Q>0$, $\forall x>0$ (INFLUENCE OF EXPECTATIONS GLOBALLY BOUNDED).

THEN THERE IS GLOBAL STABILITY FOR ALL INITIAL CONDITIONS ($x_t \rightarrow 0$, $\beta_t \rightarrow \bar{\beta}$ WITH MAY BE $\bar{\beta} \neq \lambda_1 = 0, \lambda_2 = 1/a$)

→ F. COMPLEX "LEARNING EQUILIBRIA"

- LOCAL INSTABILITY DOES NOT IMPLY GLOBALLY EXPLOSIVE DYNAMICS. IN SMOOTH LEARNING CASE, LOCAL INSTABILITY MAY IMPLY CONVERGENCE TO MEDIUM RANGE, MORE OR LESS COMPLEX "LEARNING EQUILIBRIA"

- POSSIBLE MECHANISM MAY LIE IN LEARNING ITSELF (BROCK AND HOMMES, 1997): IF CHOICE OF EFFICIENT PREDICTORS IS COSTLY, MOST AGENTS MAY CHOOSE EASY RULES OF THUMB (EG NAIVE EXPECTATIONS) NEAR A STEADY STATE WHEN ERRORS ARE SMALL THAT MAY IMPLY LOCAL INSTABILITY. MOST AGENTS MAY CHOOSE MORE EFFICIENT (EVEN THOUGH MORE COSTLY) PREDICTORS FAR AWAY FROM STEADY STATE WHEN ERRORS MAY BE LARGER (⇒ MORE COSTLY), THAT MAY MAKE GLOBAL LEARNING DYNAMICS "NON EXPLOSIVE".

⟹ MAY THEN EXIST "LEARNING EQUILIBRIA" IN BETWEEN (CYCLES OR MORE COMPLEX)

• NAIVE VERSUS RATIONAL EXPECTATIONS IN COBWEB MODEL (BROCK AND HOMMES)

• STANDARD COBWEB

DEMAND $\quad D(p_{t+1}) = A - Bp_{t+1} \quad A, B > 0$

SUPPLY $\quad S(p^e_{t+1}) = b\,p^e_{t+1} \quad b > 0$

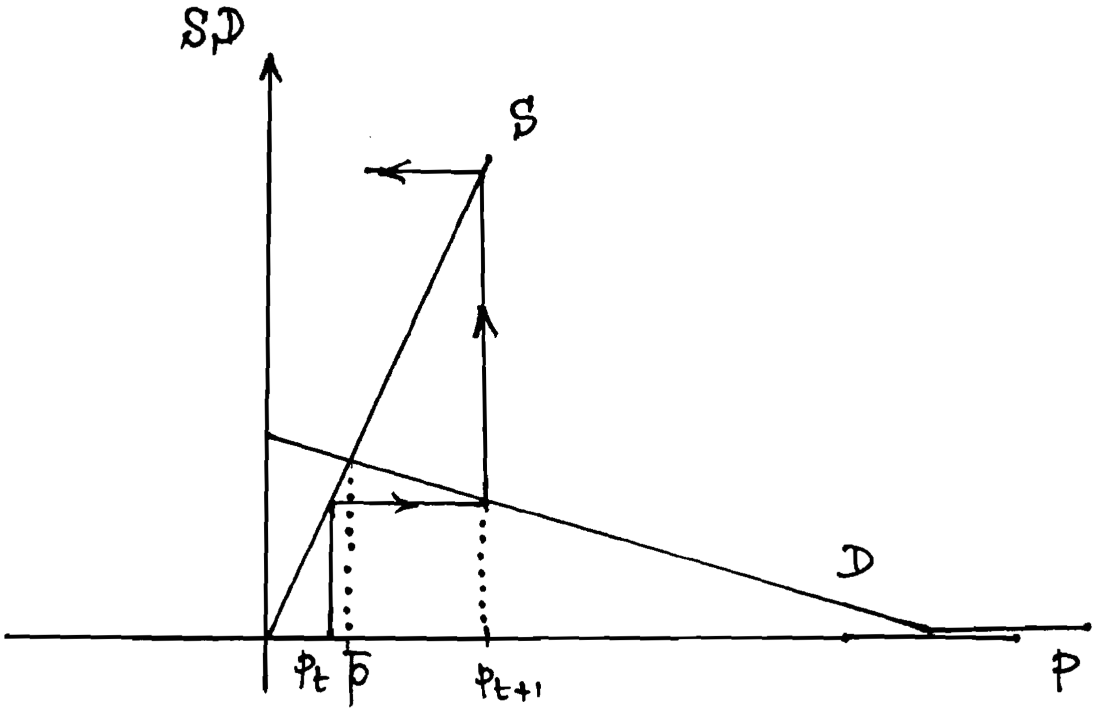
• RATIONAL EXPECTATIONS $\Rightarrow$ "HYPERSTABLE DYNAMICS"

$D(p_{t+1}) = S(p_{t+1}) \Leftrightarrow p_{t+1} = \bar{p} = A/(B+b)$ (STEADY STATE)

• NAIVE EXPECTATIONS $p^e_{t+1} = p_t$

$D(p_{t+1}) = S(p_t) \Leftrightarrow (p_{t+1} - \bar{p}) = -\frac{b}{B}(p_t - \bar{p})$

COBWEB DYNAMICS UNDER NAIVE EXPECTATIONS IS UNSTABLE IF EXPECTATIONS MATTER MUCH, $b > B$



• <u>CHOICE OF PREDICTORS</u>: n_t = PROPORTION OF AGENTS USING RATIONAL EXPECTATIONS (COSTLY PREDICTOR), $1-n_t$ = PROPORTION USING NAIVE FORECASTS (CHEAP PREDICTOR)

$$D(p_{t+1}) = n_t S(p_{t+1}) + (1-n_t) S(p_t)$$

$$\Leftrightarrow \quad (p_{t+1} - \bar{p})(B + n_t b) + (p_t - \bar{p})(1-n_t) b = 0$$

PLUS n_t = (NON LINEAR) FUNCTION OF RELATIVE PAST PERFORMANCE OF TWO PREDICTORS

$\Longrightarrow$ IF $(p_t - \bar{p})$ SMALL, $n_t \sim 0 \Rightarrow$ LOCAL DYNAMICS $\sim$ NAIVE FORECASTS $\Longrightarrow$ LOCALLY UNSTABLE IF $b > B$

$\Longrightarrow$ IF $|p_t - \bar{p}|$ LARGE, $n_t \sim 1 \Rightarrow$ GLOBAL DYNAMICS $\sim$ RATIONAL EXPECTATIONS (TAKING INTO ACCOUNT n_t)

$\Longrightarrow \quad p_{t+1} - \bar{p} = -(p_t - \bar{p})(1-n_t) b / (B + n_t b)$ SMALL IF $n_t \to 1$ FAST (IF AGENTS REACT FAST) $\Rightarrow$ NON EXPLOSIVE

$\Longrightarrow$ <u>EXISTENCE OF CYCLES OR COMPLEX ATTRACTING EQUILIBRIA IN BETWEEN</u> (GLOBAL HOMOCLINIC BIFURCATIONS)