NONLINEAR ENDOGENOUS FLUCTUATIONS: CYCLES AND SUNSPOTS

- EXOGENOUS SHOCKS TO FUNDAMENTALS VERSUS SHOCKS TO EXPECTATIONS
- MULTIPLE EQUILIBRIA: INDETERMINACY, CYCLES, STOCHASTIC ENDOGENOUS FLUCTUATIONS (SUNSPOTS)
- NEAR UNIT ROOTS: LOCAL BIFURCATIONS
- INDETERMINACY OF RESPONSE TO POLICY MOVES
- EXPECTATIONS COORDINATION PROBLEM (POSSIBLE IMPORTANT ROLE FOR POLICY)

FRAMEWORK

(TWO DIMENSIONAL DYNAMICS WITH ONE PREDETERMINED VARIABLE)

- DETERMINISTIC DYNAMICS

$$k_t = g_1(k_{t-1}, x_t), \quad x_{t+1} = g_2(k_{t-1}, x_t)$$

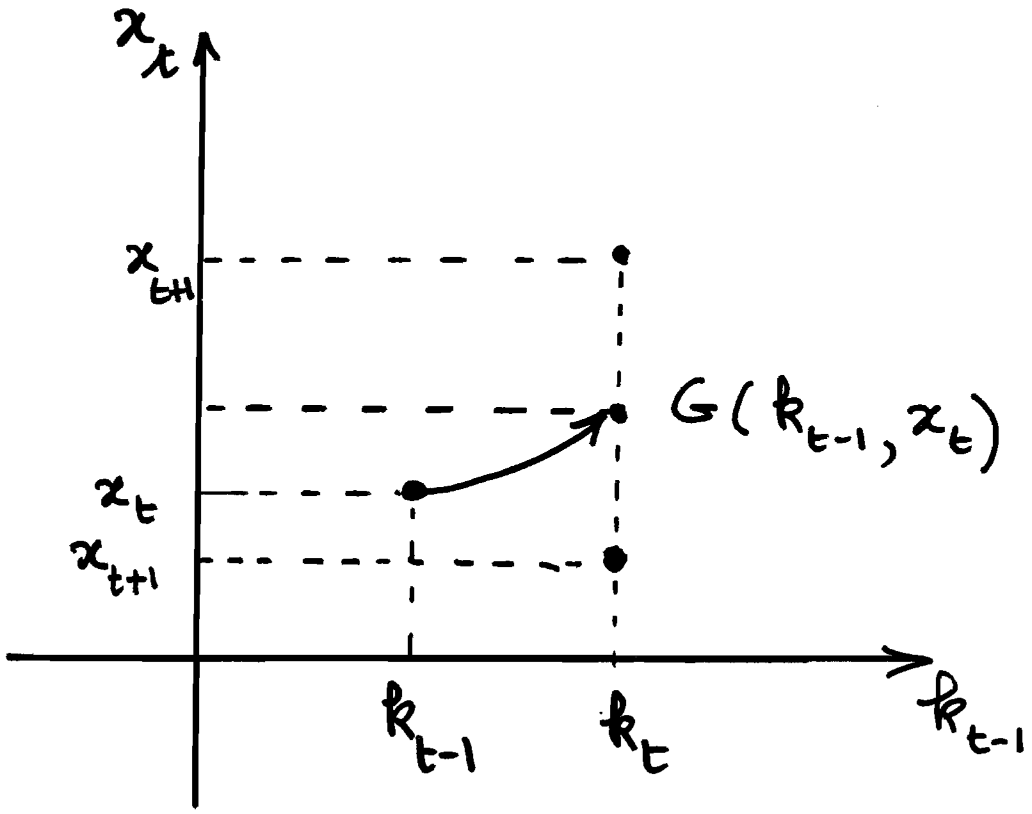
OR $\quad (k_t, x_{t+1}) = G(k_{t-1}, x_t)$

- SUNSPOT EQUILIBRIA

$$k_t = g_1(k_{t-1}, x_t), \quad x_{t+1} = g_2(k_{t-1}, x_t, \varepsilon_{t+1})$$

WITH $\quad E_t[\varepsilon_{t+1}] = 0$

$\{\varepsilon_t\}$ = SHOCKS TO EXPECTATIONS



I = SUPPORT OF STOCHASTIC SUNSPOT EQUILIBRIUM

$$G(I) \subset \underset{V}{co}\, I$$

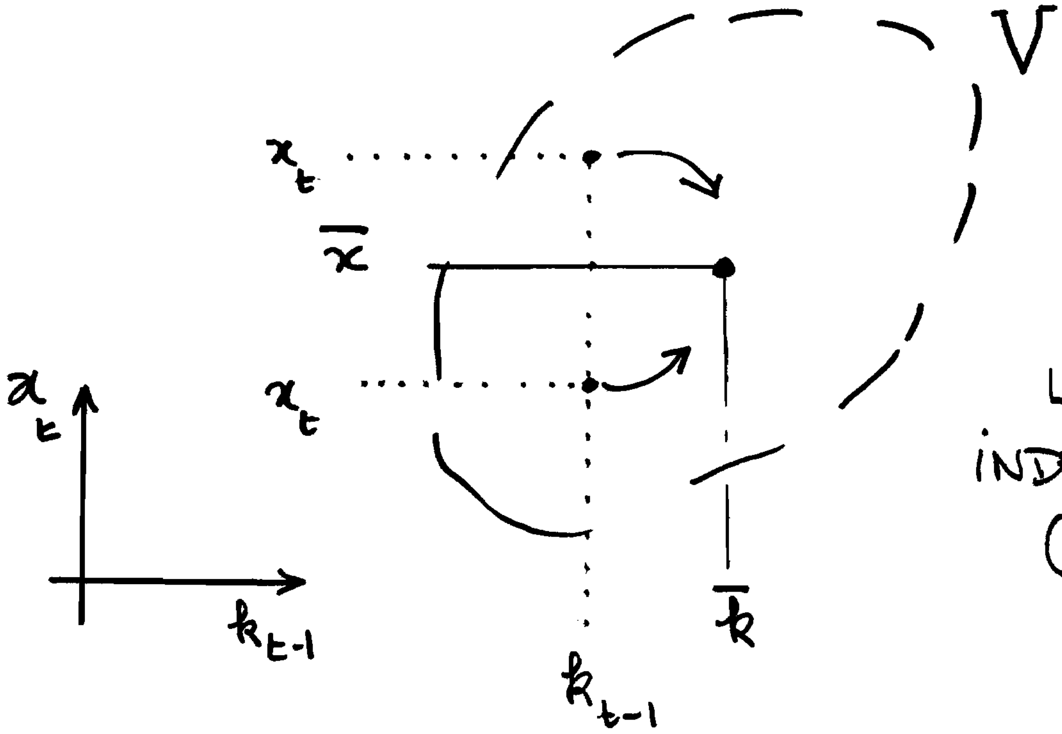
- GEOMETRICAL CHARACTERIZATION OF SUPPORT I OF SUNSPOT EQUILIBRIUM?

- NECESSARY CONDITION: THE IMAGE OF I IN DETERMINISTIC DYNAMICS, $G(I)$, IS CONTAINED IN THE "VERTICAL CONVEX HULL" OF I

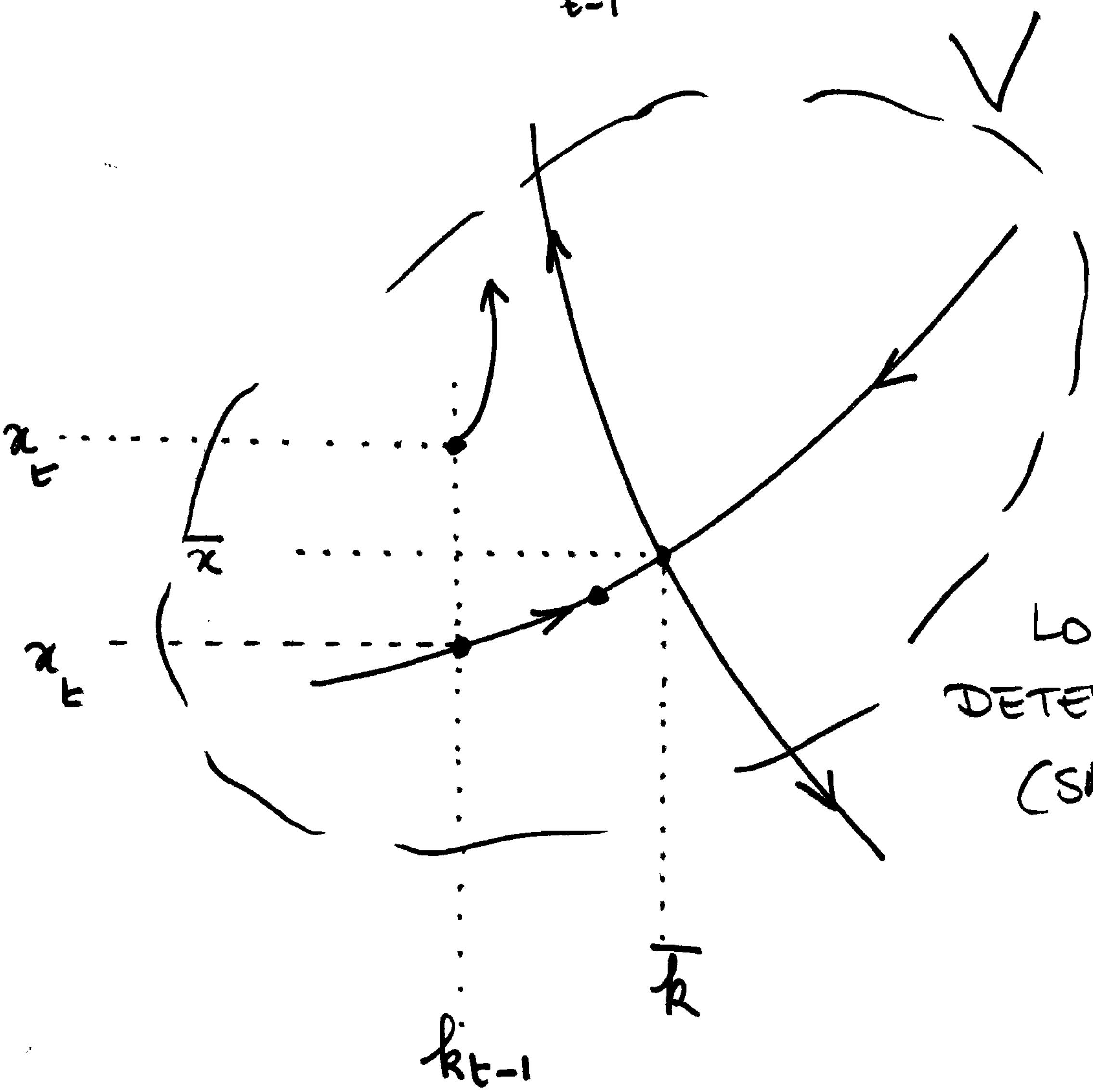
$$G(I) \subset Co_V I$$

- SUFFICIENT CONDITION: IF $G(I)$ IS CONTAINED IN THE INTERIOR OF $Co_V I$, I.E. $G(I) \subset \mathrm{Int}\, Co_V I$, THEN THERE EXISTS INFINITY OF SUNSPOT EQUILIBRIA THAT ARE NONDEGENERATE (TRULY RANDOM) AND THAT STAY IN I FOR ALL $t \geqq 1$

TRUE IN PARTICULAR IF $G(I) \subset \mathrm{Int}\, I$



LOCAL INDETERMINACY (SINK)



LOCAL DETERMINACY (SADDLE)

• CONSEQUENCES

• IF STEADY STATE IS LOCALLY INDETERMINATE (HERE: STABLE) IN DETERMINISTIC DYNAMICS, THERE EXISTS INFINITELY MANY NONDEGENERATE SUNSPOT EQUILIBRIA IN EVERY NEIGHBORHOOD OF STEADY STATE

• IF STEADY STATE IS LOCALLY DETERMINATE (HERE: SOURCE OR SADDLE) IN DETERMINISTIC DYNAMICS, THEN THERE EXISTS A NEIGHBORHOOD SUCH THAT EVERY SUNSPOT EQUILIBRIUM CONTAINED IN THAT NEIGHBORHOOD IS NECESSARILY DEGENERATE, I.E. NONRANDOM

• CHARACTERIZATION IS GLOBAL: ALLOWS TO DESCRIBE WHAT HAPPENS ALONG BIFURCATIONS

IN PARTICULAR:

- IF A LOCALLY DETERMINATE STEADY STATE (SADDLE) IS CLOSE TO A LOCALLY INDETERMINATE STEADY STATE (SINK), THERE ARE STATIONARY SUNSPOTS IN EVERY NEIGHBORHOOD OF THE LOCALLY INDETERMINATE STEADY STATE (EX: SADDLE NODE OR TRANSCRITICAL BIFURCATION)
- IF A LOCALLY DETERMINATE STEADY STATE (SADDLE) IS CLOSE TO A LOCALLY INDETERMINATE CYCLE OF PERIOD TWO (SINK), THERE ARE STATIONARY SUNSPOTS IN EVERY NEIGHBORHOOD OF THE STABLE CYCE (EX: FLIP BIFURCATION)
- IF A LOCALLY DETERMINATE STEADY STATE (SOURCE) IS CLOSE TO A LOCALLY INDETERMINATE INVARIANT CLOSED CURVE (SINK), THERE ARE STATIONARY SUNSPOTS IN EVERY NEIGHBORHOOD OF THE STABLE INVARIANT CLOSED CURVE (EX: HOPF BIFURCATION)

QUALITATIVE CONCLUSIONS:

- LINEAR UNIT ROOT MODELS ARE "STRUCTURALLY UNSTABLE": VERY DIFFERENT BEHAVIOR IF SMALL NON-LINEARITIES

- LOCAL INSTABILITY DOES NOT MEAN NECESSARILY EXPLOSIVE BEHAVIOR: STABLE NON-LINEAR DETERMINISTIC OR STOCHASTIC ENDOGENOUS FLUCTUATIONS

- POSSIBLE "CORRIDOR" EFFECTS (STABILITY FOR SMALL SHOCKS, INSTABILITY FOR LARGER SHOCKS)

I. APPLICATION : NONLINEAR ENDOGENOUS FLUCTUATIONS AND CAPITAL LABOR SUBSTITUTION (GEOMETRICAL METHODS)

(GRANDMONT, PINTUS, DE VILDER, JET 1998)

- <u>MODEL</u>: WOODFORD (JET, 1986). INFINITE HORIZON, TWO ASSETS (MONEY, PHYSICAL CAPITAL)

"<u>WORKERS</u>": LIQUIDITY (CASH IN ADVANCE) CONSTRAINT

"<u>CAPITALISTS</u>": NO LIQUIDITY CONSTRAINT, SMALL DISCOUNT RATE $\Rightarrow$ HOLD THE WHOLE CAPITAL STOCK, NOT MONEY (DOMINATED ASSET NEAR STEADY STATE)

$\Rightarrow$ <u>2D DYNAMICS SIMILAR TO OLG MODEL</u>, AT LEAST <u>LOCALLY</u>, I.E. NEAR STEADY STATE

- <u>WORKERS</u>: $\max E_t[V_2(c_{t+1})] - V_1(\ell_t)$ S.T.

$$p_{t+1} c_{t+1} = m_t^d = w_t \ell_t$$

$$\Rightarrow \quad v_1(\ell_t) = E_t v_2(c_{t+1}) \quad \text{WITH} \quad \begin{cases} v_1(\ell) = \ell V_1'(\ell) \\ v_2(c) = c V_2'(c) \end{cases}$$

- <u>CAPITALISTS</u> (COBB DOUGLAS, DISCOUNT FACTOR $\beta \simeq 1$)

$$\Rightarrow \quad k_t = \beta R_t k_{t-1} \simeq R_t k_{t-1}$$

• **PRODUCERS**: CONSTANT RETURNS

$(k_{t-1}, \ell_t) \longrightarrow y_t = \ell_t f(a_t), \ a_t = k_{t-1}/\ell_t$

MARGINAL PRODUCTIVITY OF LABOR: $\omega(a_t)$
MARGINAL PRODUCTIVITY OF CAPITAL: $\rho(a_t)$
GROSS RATE OF RETURN OF CAPITAL
$R(a_t) = 1 + \rho(a_t) - \delta, \ \delta =$ DEPRECIATION RATE

• **COMPETITIVE INTERTEMPORAL EQUILIBRIUM**

– **WITH SUNSPOTS**: SEQUENCE OF RANDOM VARIABLES (k_{t-1}, a_t) SATISFYING FOR ALL $t \geqq 1$

$$k_t = R(a_t) k_{t-1}, \ v_1(k_{t-1}/a_t) = E_t\left[v_2(k_t \omega(a_{t+1})/a_{t+1})\right]$$

OR $v_2(k_t \omega(a_{t+1})/a_{t+1}) = v_1(k_{t-1}/a_t) + \varepsilon_{t+1}, \ E_t[\varepsilon_{t+1}] = 0$

– **DETERMINISTIC**:

$$k_t = R(a_t) k_{t-1}, \quad \omega(a_{t+1})/a_{t+1} = \gamma(k_{t-1}/a_t)/k_t$$

WITH $\gamma = v_2^{-1} \circ v_1$ ("OFFER CURVE")

$\Rightarrow$ TWO DIMENSIONAL DYNAMICS $(k_t, a_{t+1}) = G(k_{t-1}, a_t)$

• **UNIQUE STEADY STATE**

→ LOCAL DYNAMICS. BIFURCATIONS

- LINEARIZATION ⇒ CHARACTERISTIC POLYNOMIAL

$$Q(z) \equiv z^2 - Tz + D = 0$$

$T = z_1 + z_2$, SUM OF ROOTS

$D = z_1 z_2$, PRODUCT OF ROOTS

- STUDY IN FUNCTION OF PARAMETERS:

δ (DEPRECIATION RATE), s (SHARE OF CAPITAL IN TOTAL INCOME), σ (CAPITAL LABOR ELASTICITY OF SUBSTITUTION) AND ε_γ (ELASTICITY OF OFFER CURVE) ($1/(\varepsilon_\gamma - 1)$ = WAGE ELASTICITY OF LABOR SUPPLY) AT STEADY STATE.

- **GEOMETRICAL METHOD:**

STUDY LOCATION OF (T,D) IN PLANE

- **BIFURCATIONS LOCI:**

$$Q(z) \equiv z^2 - Tz + D = 0$$

$Q(1) = 1 - T + D = 0$ (LINE (AC): SADDLE NODE OR TRANSCRITICAL)

$Q(-1) = 1 + T + D = 0$ (LINE (AB): FLIP)

$D = 1 \ , |T| \leq 2$ (SEGMENT [BC]: HOPF)

- $T = T_1 - \dfrac{\varepsilon_\gamma - 1}{\varepsilon_\omega - 1}$, WITH $T_1 = 1 + \dfrac{|\varepsilon_R| - 1}{\varepsilon_\omega - 1}$

$D = D_1 - (\varepsilon_\gamma - 1)\dfrac{|\varepsilon_R| - 1}{\varepsilon_\omega - 1}$ WITH $D_1 = \dfrac{|\varepsilon_R| - 1}{\varepsilon_\omega - 1}$

$\varepsilon_\omega = s/\sigma \quad , \quad |\varepsilon_R| = \delta(1-s)/\sigma$

$\Rightarrow$ (T,D) DESCRIBES HALF LINE $\Delta(\sigma)$, FROM (T_1, D_1) ON (AC) WHEN $\varepsilon_\gamma = 1$ (INFINITE WAGE ELASTICITY OF LABOR SUPPLY) TO INFINITY WHEN $\varepsilon_\gamma \to +\infty$ (ZERO WAGE ELASTICITY OF LABOR SUPPLY)

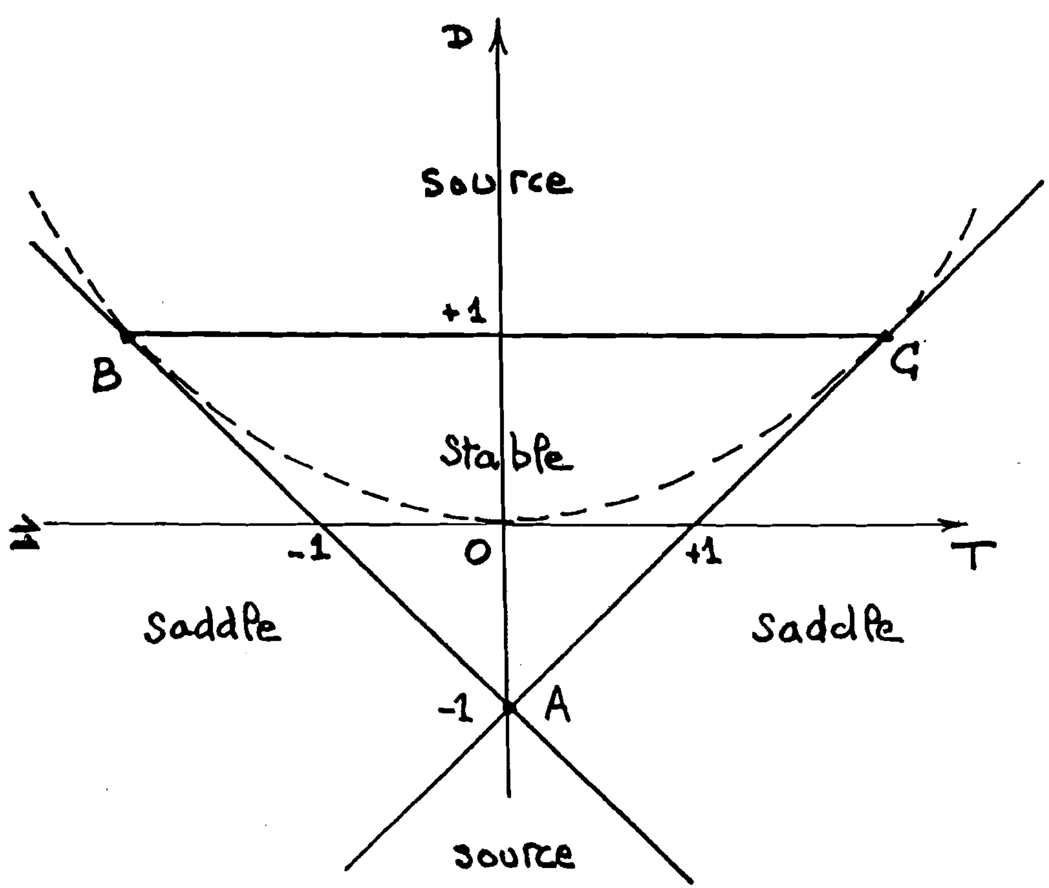

D
Source
+1
B
C
Stable
-1
0
+1
T
Saddle
Saddle
-1
A
Source

Fig. 1.a

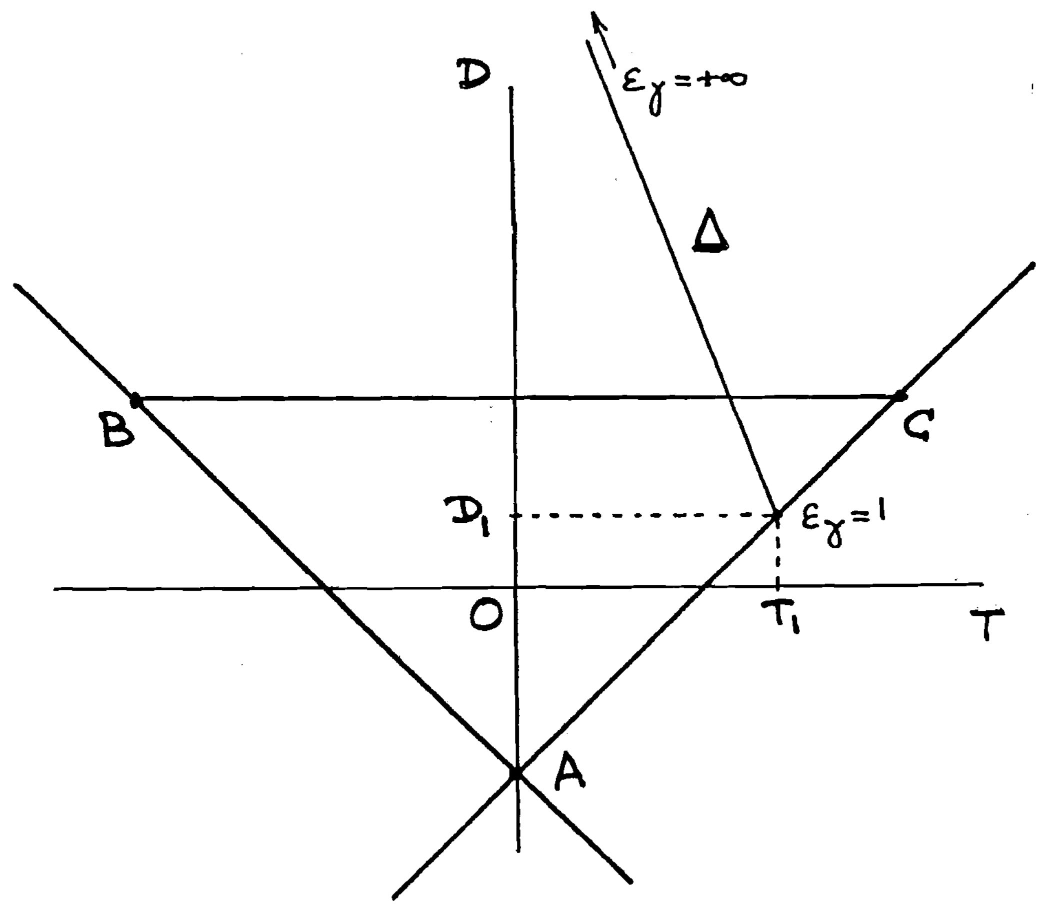

D
$\varepsilon_\gamma = +\infty$
Δ
B
C
D_1
$\varepsilon_\gamma = 1$
O
T_1
T
A

Fig. 1.b

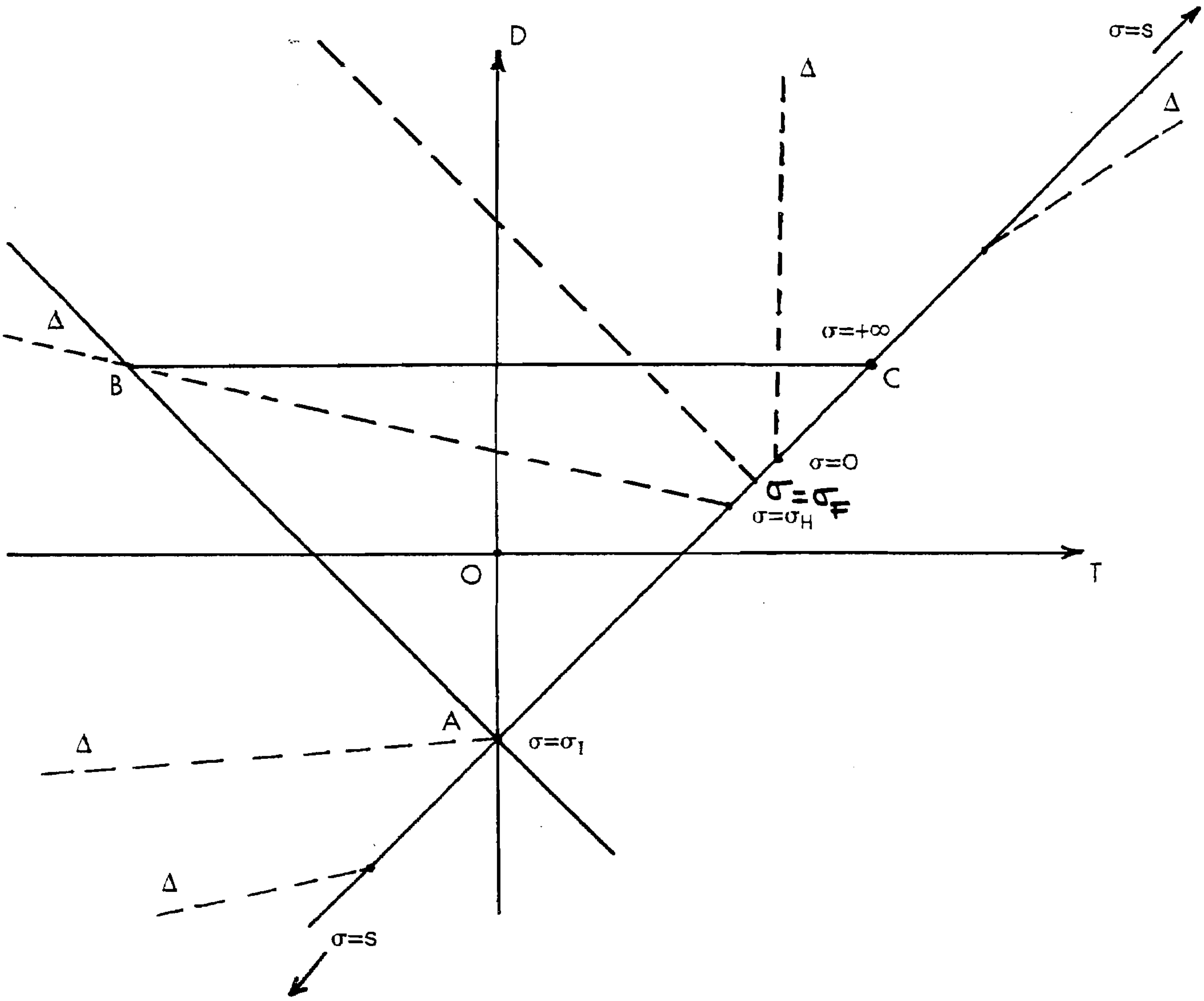


FIGURE 1

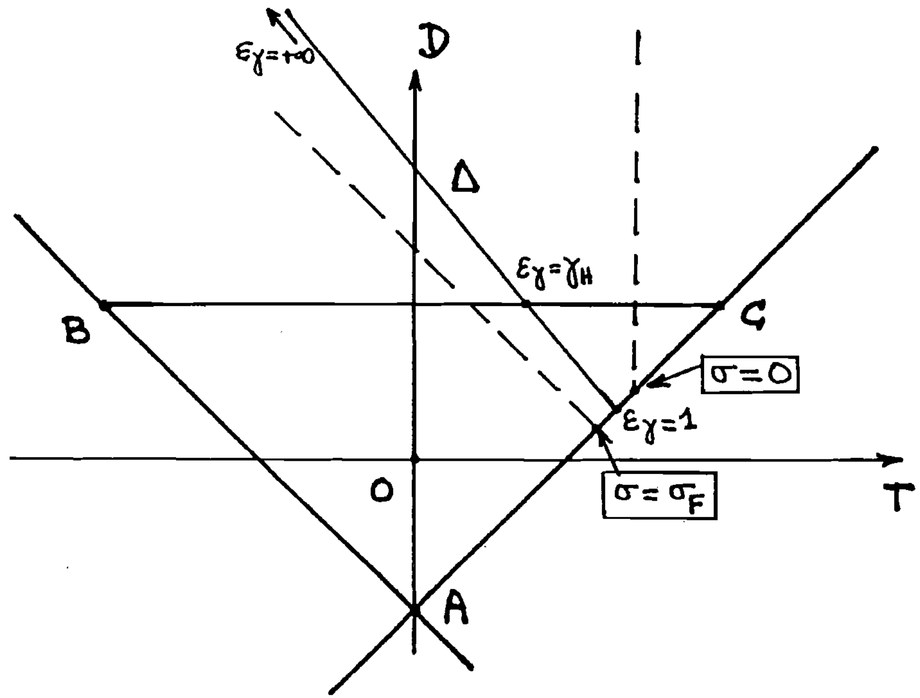


Fig. 2.a

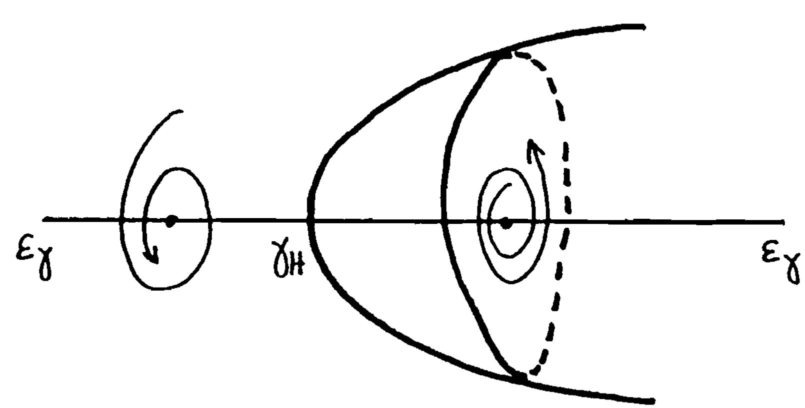


Fig. 2.b

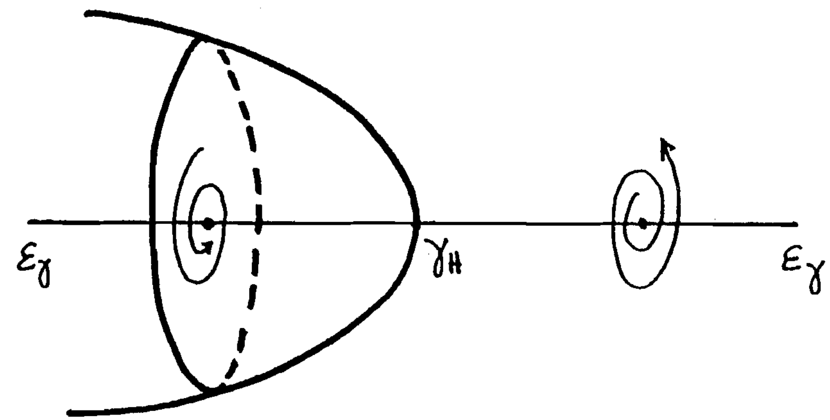


Fig. 2.c

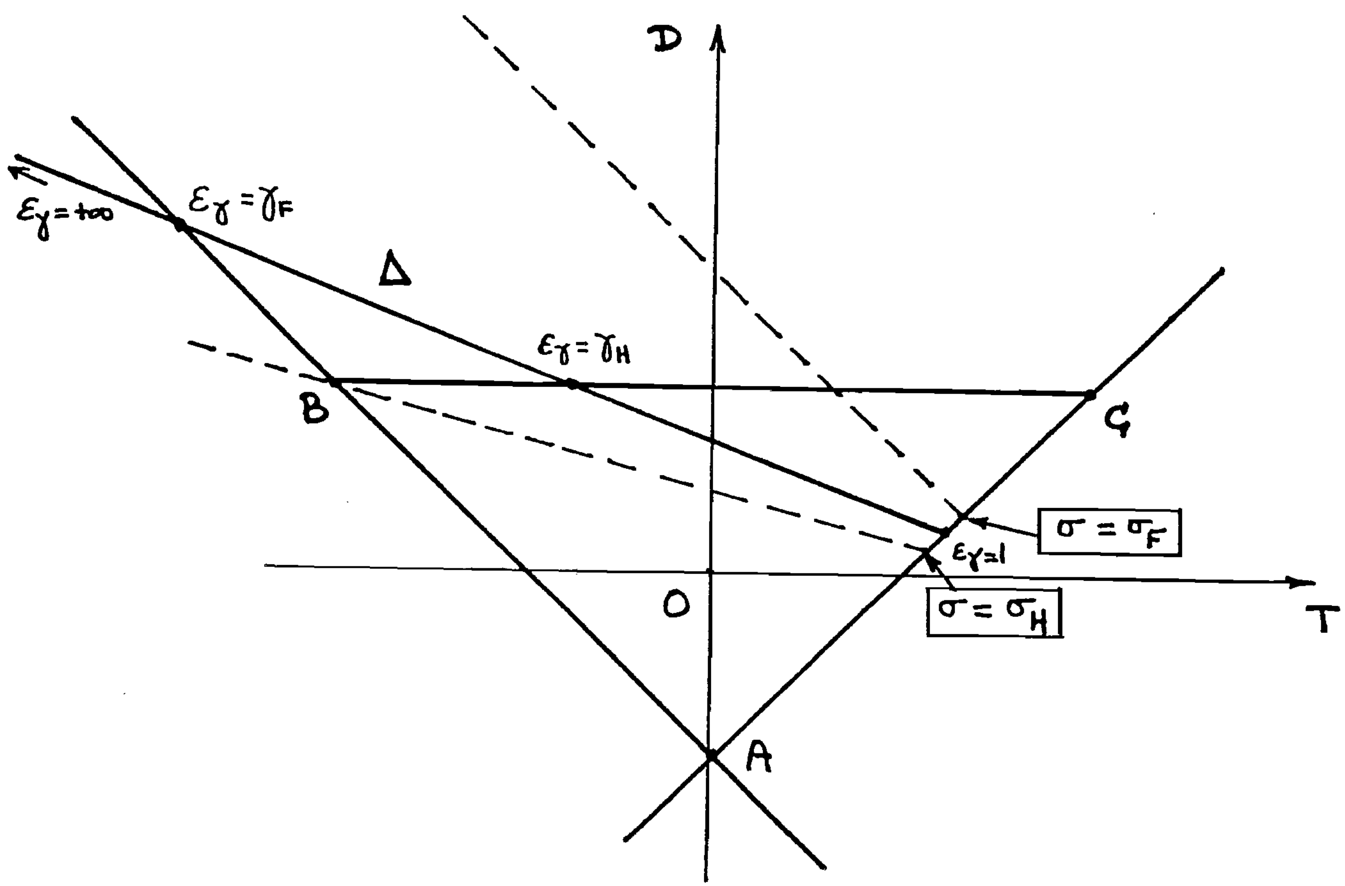

D
T
O
A
B
C
Δ
$\varepsilon_\gamma = +\infty$
$\varepsilon_\gamma = \gamma_F$
$\varepsilon_\gamma = \gamma_H$
$\varepsilon_\gamma = 1$
$\sigma = \sigma_F$
$\sigma = \sigma_H$

Fig. 3

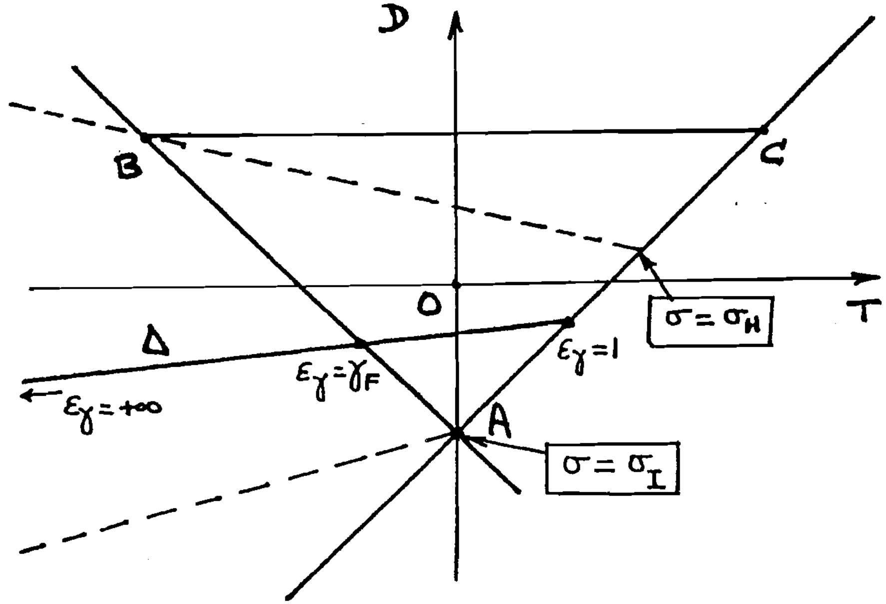


Fig. 4.a

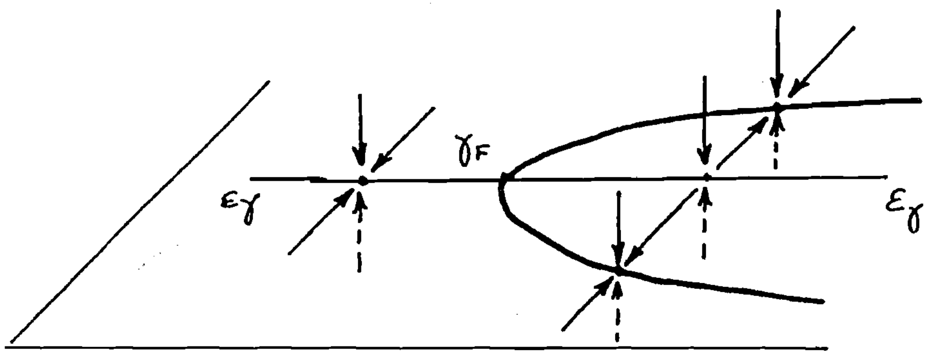


Fig. 4.b

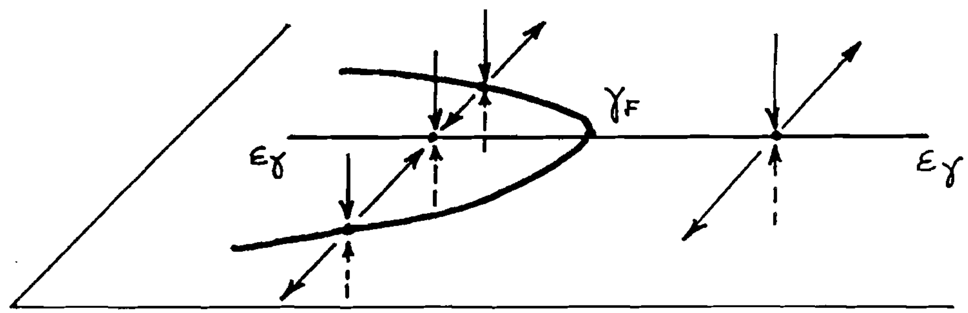


Fig. 4.c

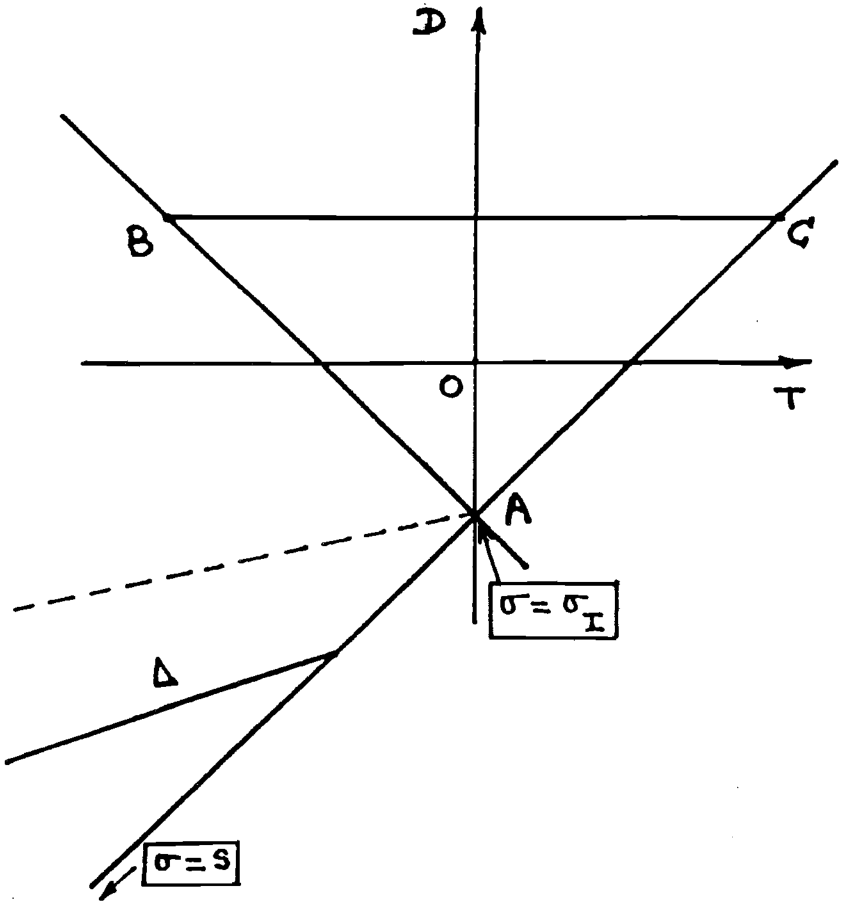


Fig. 5.a

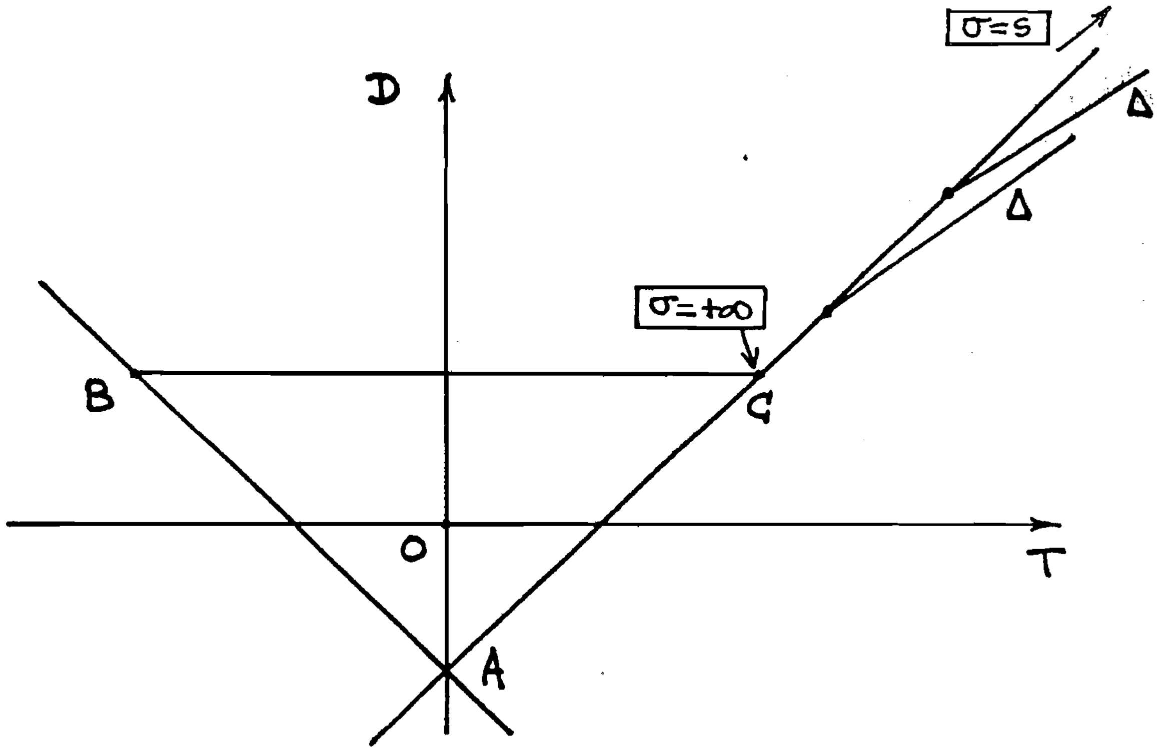
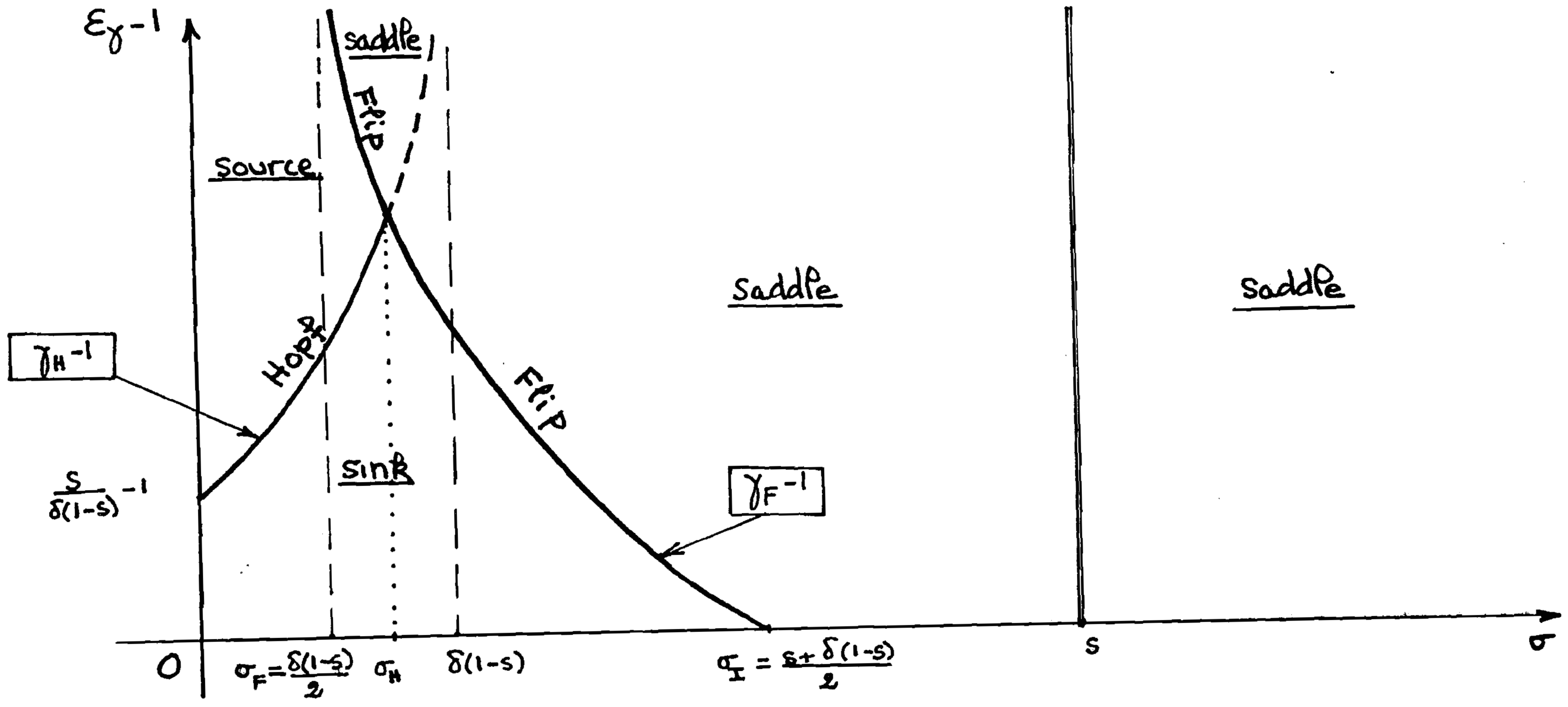


Fig. 5.b



$\varepsilon_\gamma - 1$
source
saddle
Flip
Hopf
sink
saddle
Flip
saddle
γ_H^{-1}
γ_F^{-1}
$\frac{s}{\delta(1-s)} - 1$
O
$\sigma_F = \frac{\delta(1-s)}{2}$
σ_H
$\delta(1-s)$
$\sigma_I = \frac{s+\delta(1-s)}{2}$
s
b

Fig. 6

→ CONSTANT ELASTICITIES

- **HOUSEHOLDS**:

$$V_1(\ell) = \ell^{1+\alpha_1}/(1+\alpha_1)\ , \quad V_2(c) = (c/B)^{1-\alpha_2}/(1-\alpha_2)$$

$\Rightarrow$ OFFER CURVE : $c = \gamma(\ell) = B\,\ell^{\gamma}$

WITH $\quad \varepsilon_\gamma = \gamma = (1+\alpha_1)/(1-\alpha_2) > 1$

- **PRODUCERS**:

$$f(a) = A\left(s\,a^{-\eta} + 1 - s\right)^{-1/\eta} \quad \text{WITH } \sigma = 1/(1+\eta)$$

- **NUMERICAL APPLICATION**: $\delta = 0.1\ , \quad s = 1/3$

$$\Rightarrow \quad \sigma_F = 3.33\%\ , \quad \sigma_H \simeq 5.1\%\ , \quad \sigma_I = 20\%$$

$$(\eta_F = 29) \qquad (\eta_H \simeq 18.6) \qquad (\eta_I = 4)$$

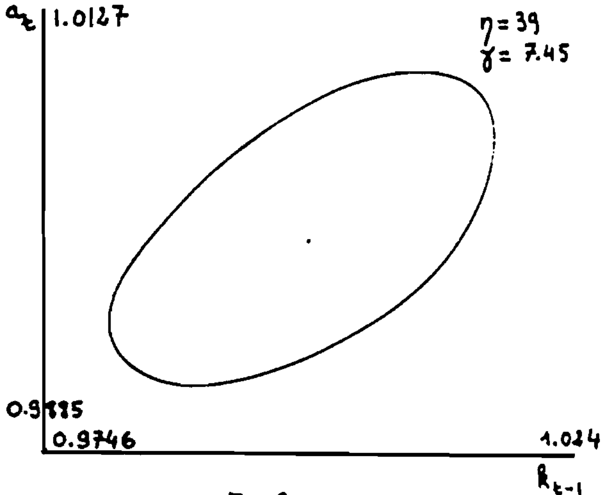


Fig. 9.a

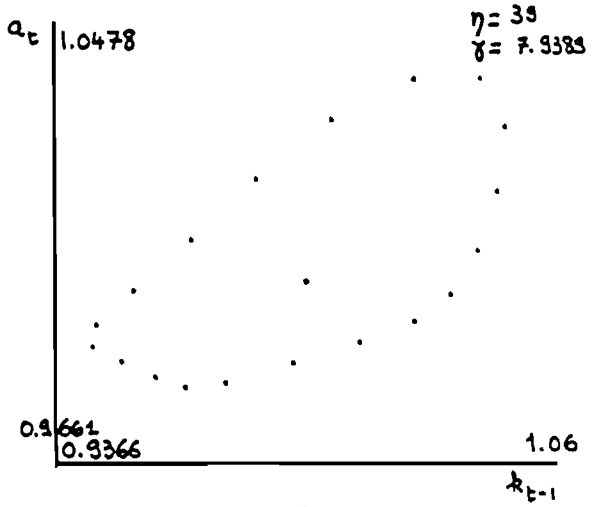


Fig. 9.b

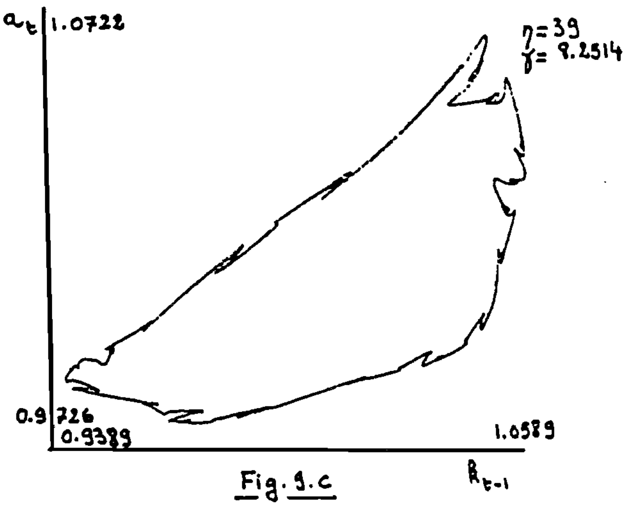


Fig. 9.c

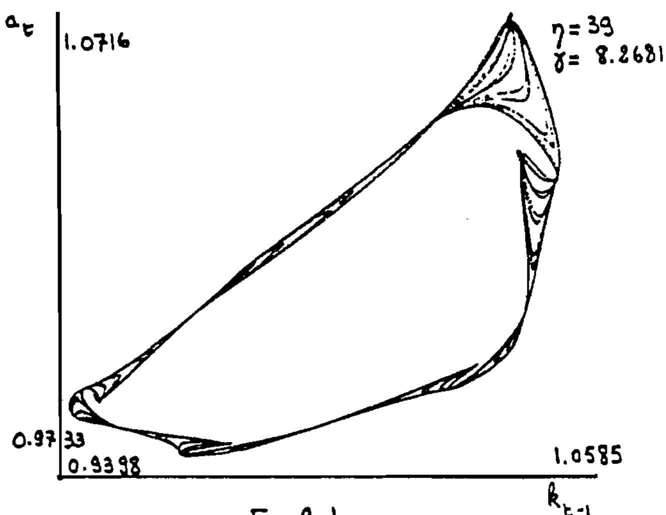


Fig. 9.d

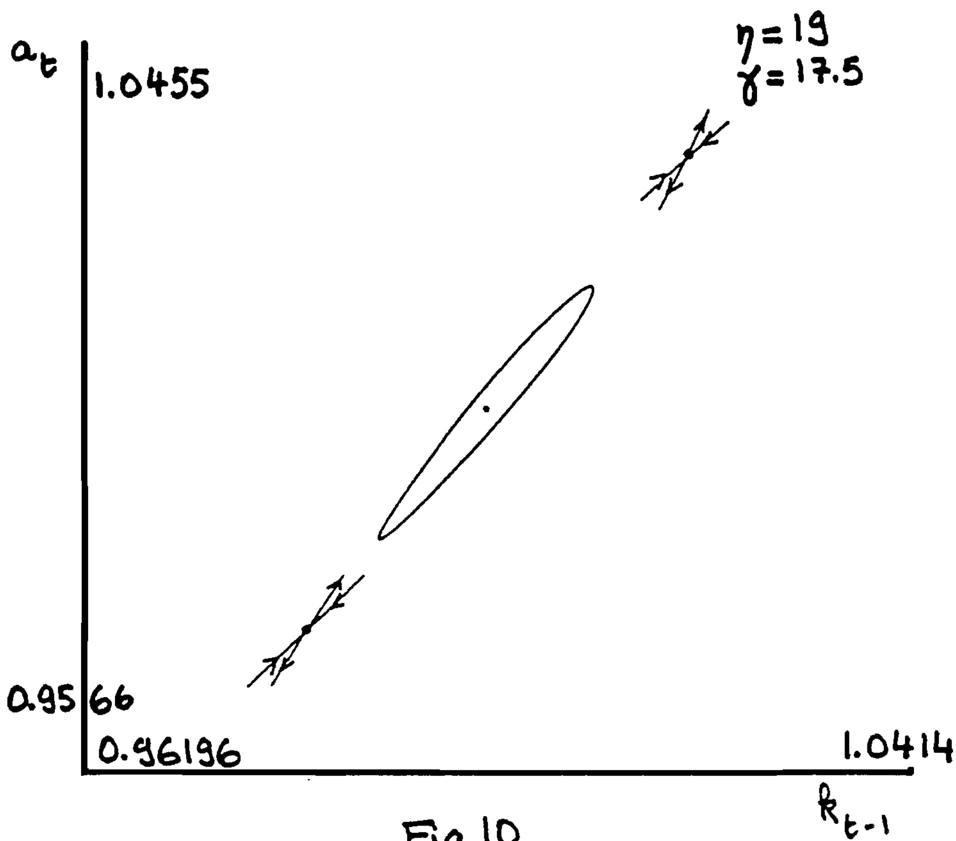


Fig. 10

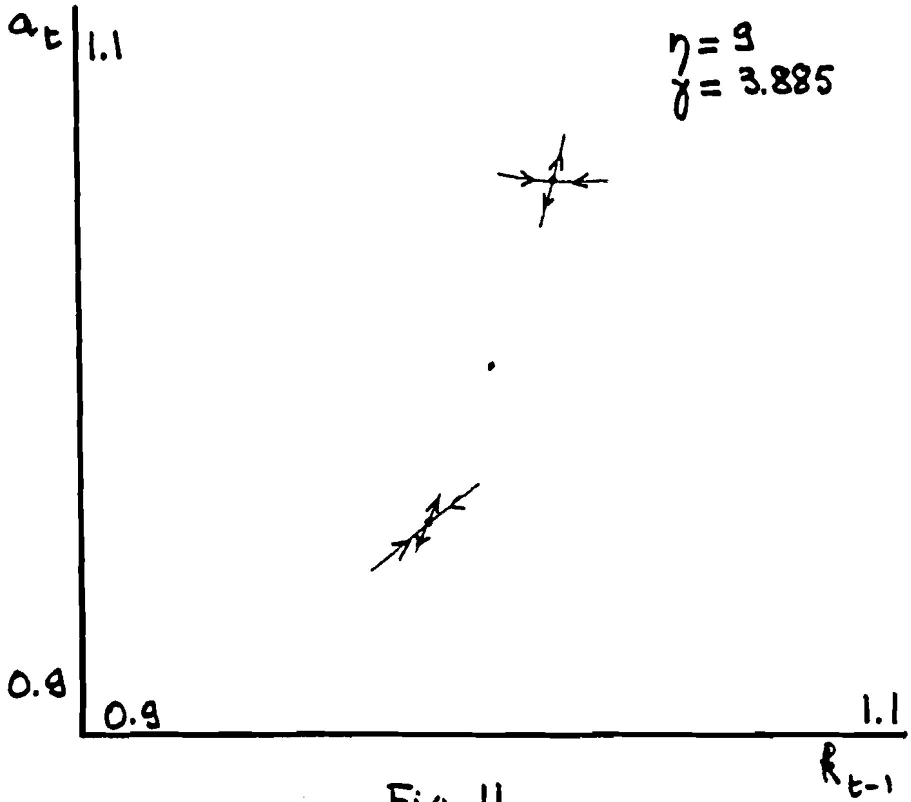


Fig. 11

→ II. APPLICATION: INCREASING RETURNS TO SCALE (CAZZAVILLAN, LLOYD-BRAGA, PINTUS, JET 1998)

- CONSTANT RETURNS, PERFECT COMPETITION (PREVIOUS MODEL): LOCAL INDETERMINACY FOR LOW ELASTICITIES OF CAPITAL LABOR SUBSTITUTION, $\sigma < \sigma_I$
- EXAMPLES IN LITERATURE OF LOCAL INDETERMINACY WITH INCREASING RETURNS (EXTERNALITIES OR INTERNAL INCREASING RETURNS WITH IMPERFECT COMPETITION) OFTEN ONLY FOR COBB-DOUGLAS PRODUCTION FUNCTION ($\sigma = 1$)
- ROBUSTNESS? SAME GEOMETRICAL METHODS

⇒ LOCAL INDETERMINACY AND HOPF FOR LOW ELASTICITIES σ OF CAPITAL LABOR SUBSTITUTION BUT ALSO FOR LARGER ELASTICITIES $\sigma > \sigma_2$

⇒ MULTIPLE PARETO RANKED STEADY STATES, TRANSCRITICAL BIFURCATIONS. "STRUCTURAL INSTABILITY" OF COBB DOUGLAS $\sigma = 1$.

• MODEL: SAME AS BEFORE + EXTERNALITIES

$$F(k,\ell) = \underbrace{A(\bar{k},\bar{\ell})}\ \underbrace{\ell f(a)}$$

PRODUCTIVE EXTERNALITY
$\bar{k}$ = AVERAGE CAPITAL
$\bar{\ell}$ = AVERAGE LABOR

SAME AS BEFORE
(CONSTANT RETURNS)
$a = k/\ell$

$$A(k,\ell) = A\,\ell^{\nu}\,\psi(a)$$

• $\varepsilon_\psi(a) = a\,\psi'(a)/\psi(a) > 0$: LOCAL MEASURE OF CAPITAL EXTERNALITY (LEARNING BY DOING)

• $\nu - \varepsilon_\psi(a) > 0$: LOCAL MEASURE OF LABOR EXTERNALITY (LEARNING BY DOING, SEARCH ON LABOR MARKET)

• ν : LOCAL MEASURE OF OVERALL EXTERNALITY (PROPORTIONAL INCREASE OF CAPITAL, LABOR)

• COMPETITIVE INTERTEMPORAL EQUILIBRIUM

SAME FOC FOR AGENTS, SAME 2D DYNAMICS WITH

$$\left|\begin{array}{l} \omega(a) \longrightarrow \ \Omega(k,\ell) = A(k,\ell)\,\omega(a) \\ R(a) \longrightarrow \ R(k,\ell) = A(k,\ell)\,\rho(a) + 1 - \delta \end{array}\right.$$

• POSSIBILITY OF MULTIPLE (TWO) PARETO RANKED STEADY STATES FOR MIDDLE RANGE EXTERNALITIES ν. "STRUCTURAL INSTABILITY" OF COBB DOUGLAS $\sigma = 1$

→ **ASSUMPTION 1**: $\varepsilon\psi(1+\delta) < \nu < \nu^*$

- ν^* IS LARGE IF PERIOD IS SHORT (δ SMALL)

- $(\nu - \varepsilon\psi)/\nu$ [RELATIVE CONTRIBUTION OF LABOR IN OVERALL EXTERNALITY] EXCEEDS $\delta/(1+\delta)$ [SMALL IF SHORT PERIOD]

⇒ Δ_1 HAS POSITION DESCRIBED IN NEXT FIGURE

→ **ASSUMPTION 2**: $\varepsilon\psi < \nu(1-s)$

- $(\nu - \varepsilon\psi)/\nu > s$

⇒ SLOPE OF HALF LINE $\Delta < 1$ FOR $\sigma = \sigma_{H2}$

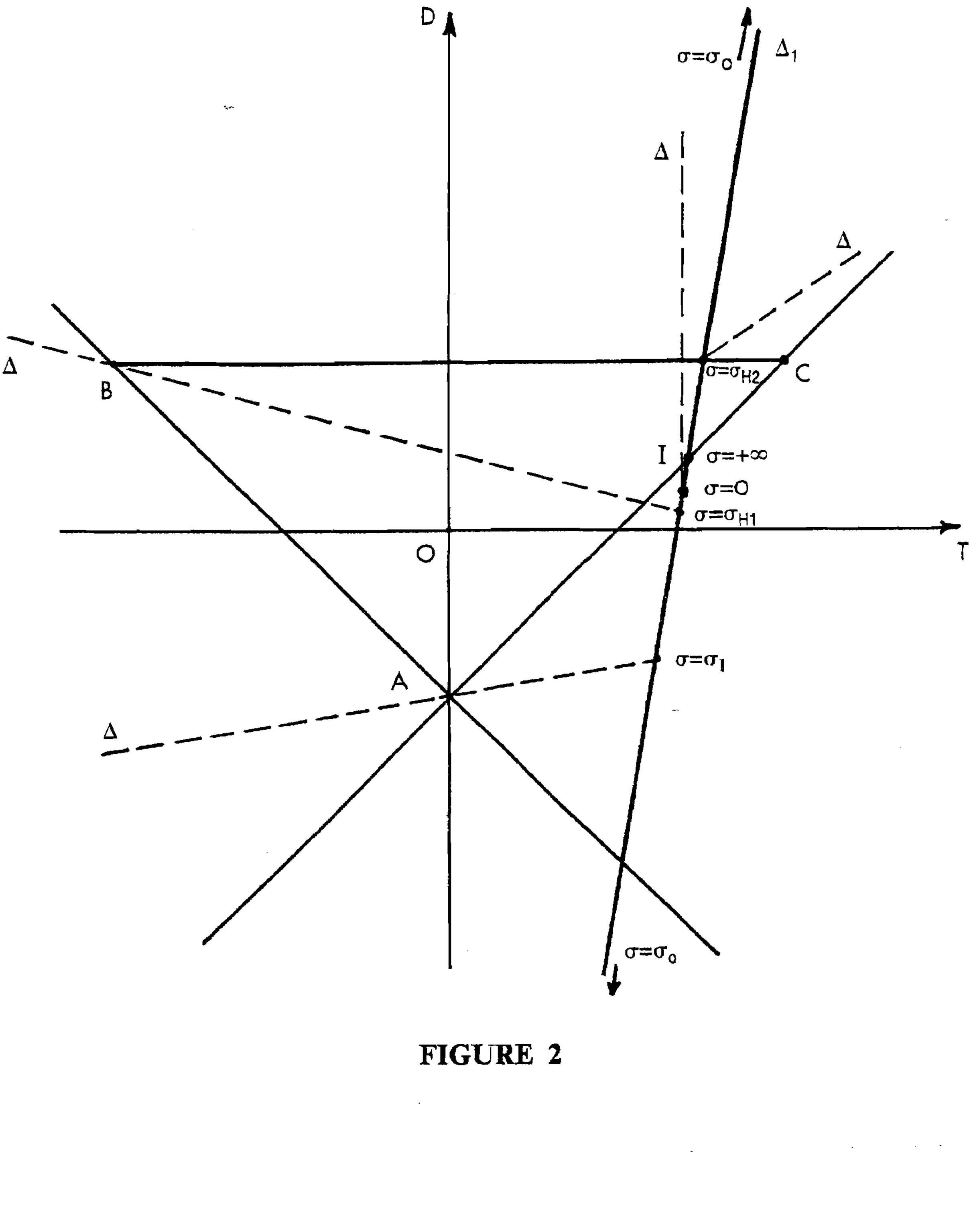
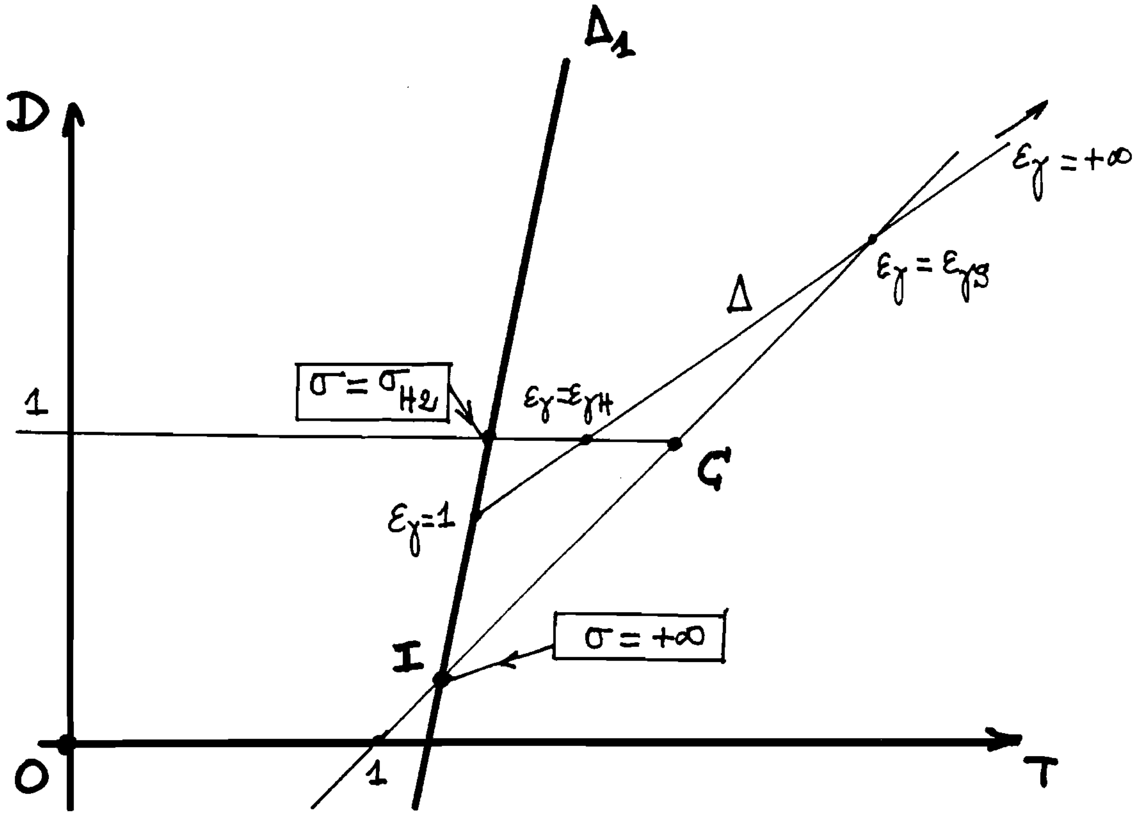
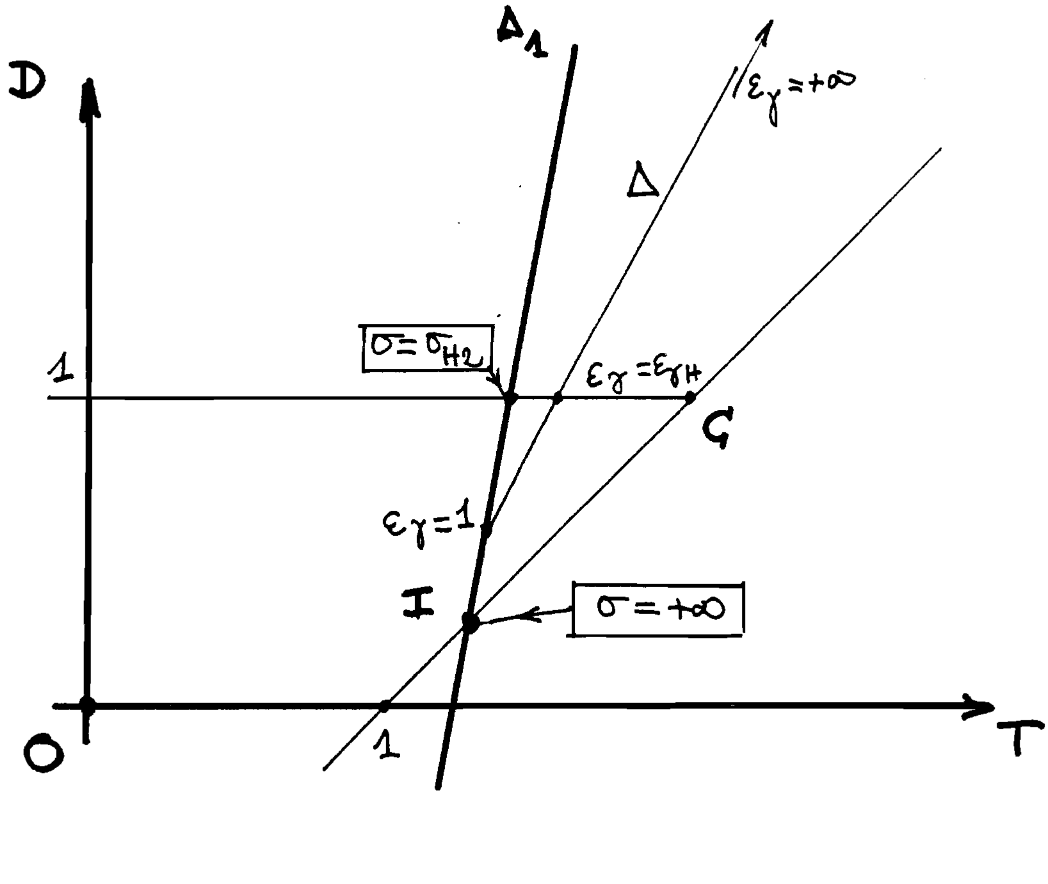


FIGURE 2



D
Δ_1
$\varepsilon_\gamma = +\infty$
$\varepsilon_\gamma = \varepsilon_{\gamma S}$
Δ
$\sigma = \sigma_{H2}$
$\varepsilon_\gamma = \varepsilon_{\gamma H}$
1
G
$\varepsilon_\gamma = 1$
$\sigma = +\infty$
I
O
1
T



D
Δ_1
$\varepsilon_\gamma = +\infty$
Δ
$\sigma = \sigma_{H2}$
$\varepsilon_\gamma = \varepsilon_{\gamma H}$
1
G
$\varepsilon_\gamma = 1$
I
$\sigma = +\infty$
O
1
T

⇒ <u>CONCLUSION</u> : UNDER ASSUMPTION 1, THERE IS LOCAL INDETERMINACY IF

$$\sigma > \sigma_{H_2} = [s - \delta(1-s)] / [\nu - (1+\delta)\varepsilon_\psi]$$

AND

$$1 < \varepsilon_\gamma < \varepsilon_{\gamma H} = [s - \sigma(1+\nu-\varepsilon_\psi)] / [\delta(1-s) - \sigma(1+\delta\varepsilon_\psi)]$$

(REMINDER: $1/(\varepsilon_\gamma - 1)$ = REAL WAGE ELASTICITY OF LABOR SUPPLY)

- EMPIRICAL RELEVANCE? $\nu, \varepsilon_\psi, \sigma_{H_2}$ NOT TOO LARGE, $\varepsilon_{\gamma H}$ NOT TOO CLOSE TO 1.

III. APPLICATION: MARKET IMPERFECTIONS

(SEEGMULLER 2004)

- INCREASING RETURNS (PREVIOUS MODEL) ⟹ SPECIFIC (SYMMETRIC) MARKET IMPERFECTIONS

$$\begin{cases} \omega(a) \longrightarrow \Omega(k,\ell) = A(k,\ell)\,\omega(a) \\ \rho(a) \longrightarrow \rho(k,\ell) = A(k,\ell)\,\rho(a) \end{cases}$$

- EXTENSION TO GENERAL (ASYMMETRIC) MARKET IMPERFECTIONS WITH ARBITRARY $\Omega(k,\ell)$, $\rho(k,\ell)$

$$\begin{cases} \varepsilon_{\Omega,K} = \alpha_{LK} + \frac{\beta_{LK}}{\sigma} + \frac{s}{\sigma} \\ \varepsilon_{\Omega,L} = \underbrace{\alpha_{LL} + \frac{\beta_{LL}}{\sigma}}_{\text{IMPERFECTION}} \underbrace{- \frac{s}{\sigma}}_{\sim \varepsilon_\omega} \end{cases} \qquad \begin{cases} \varepsilon_{\rho,K} = \alpha_{KK} + \frac{\beta_{KK}}{\sigma} - \frac{1-s}{\sigma} \\ \varepsilon_{\rho,L} = \underbrace{\alpha_{KL} + \frac{\beta_{KL}}{\sigma}}_{\text{IMPERFECTION}} \underbrace{+ \frac{1-s}{\sigma}}_{\sim \varepsilon_\rho} \end{cases}$$

- COVERS IMPERFECT COMPETITION WITH MARKUP VARIABILITY, TASTE FOR VARIETY; FISCAL POLICY AND BALANCED BUDGET RULES; GENERAL ASYMMETRIC EXTERNALITIES

$$F(k,\ell) = AF(C(k,\ell)\,k,\ D(k,\ell)\,\ell)$$

IV. APPLICATION : NON MONETARY OLG MODEL

(LLOYD-BRAGA, NOURRY, VENDITTI, JET 2006)

- TWO PERIODS OLG: MAX $U(c_t, c_{t+1}) - V(\ell_t)$

S.T. $c_t + k_t = \omega_t \ell_t$, $c_{t+1} = R_{t+1} k_t$

U HOMOGENOUS DEGREE 1 $\Rightarrow c_t = \alpha(R_{t+1}) \omega_t \ell_t$

WHERE $\alpha(R)$ = PROPENSITY TO CONSUME OF THE YOUNG AND $\varepsilon_\alpha(R) = (1-\alpha(R))(1-\gamma(R)) < 0$ IFF THE ELASTICITY OF INTERTEMPORAL SUBSTITUTION $\gamma(R) > 1$

- PRODUCTION EXTERNALITIES

$$F(k_{t-1}, \ell_t) = A(k_{t-1}, \ell_t)\, \ell_t f(a_t), \quad a_t = k_{t-1}/\ell_t$$

$\Longrightarrow$ 2D DYNAMICS $(k_{t-1}, a_t) \rightarrow (k_t, a_{t+1})$

- A STEADY STATE IS LOCALLY DETERMINATE IF NO EXTERNALITIES, OR IF $\varepsilon_{A,k} \geq (1-\alpha)\varepsilon_{A,\ell}$

$\Longrightarrow$ FOCUS ON NEGLIGIBLE CAPITAL EXTERNALITIES $\varepsilon_{A,k} = 0 < \varepsilon_{A,\ell}$.

- **LINEARIZED LOCAL DYNAMICS NEAR A STEADY STATE** :

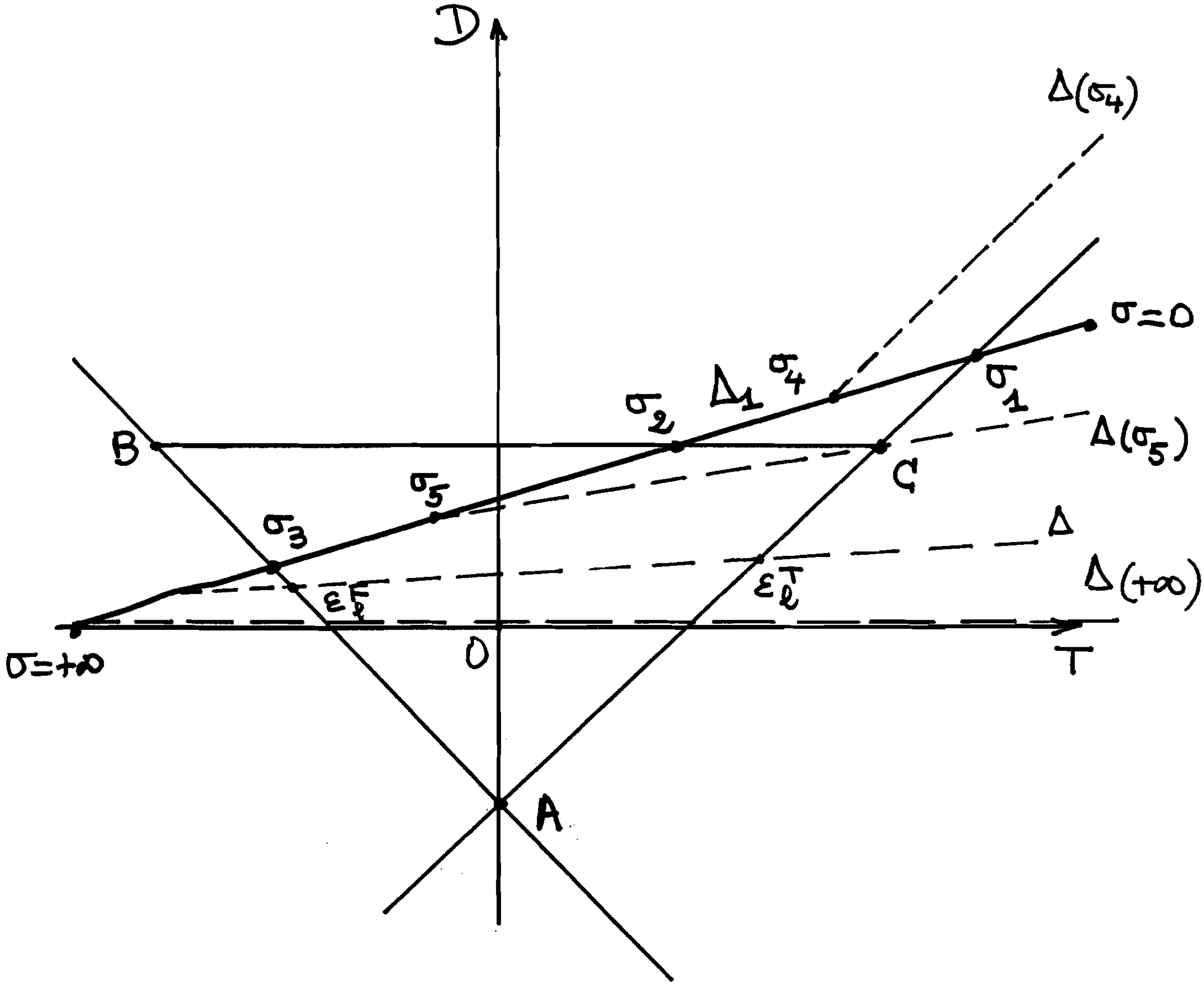
(D,T) VARIES LINEARLY WITH $(1+\varepsilon_\ell)/\varepsilon_\ell$ ($\sim$ ELASTICITY ε_γ OF OFFER CURVE IN PREVIOUS MODELS) WITH ε_ℓ = WAGE ELASTICITY OF LABOR SUPPLY

$\Rightarrow$ (D,T) DESCRIBES HALF LINE $\Delta(\sigma)$ FROM (D_1, T_1) FOR $\varepsilon_\ell \to +\infty$ TO INFINITY WHEN $\varepsilon_\ell \searrow 0$

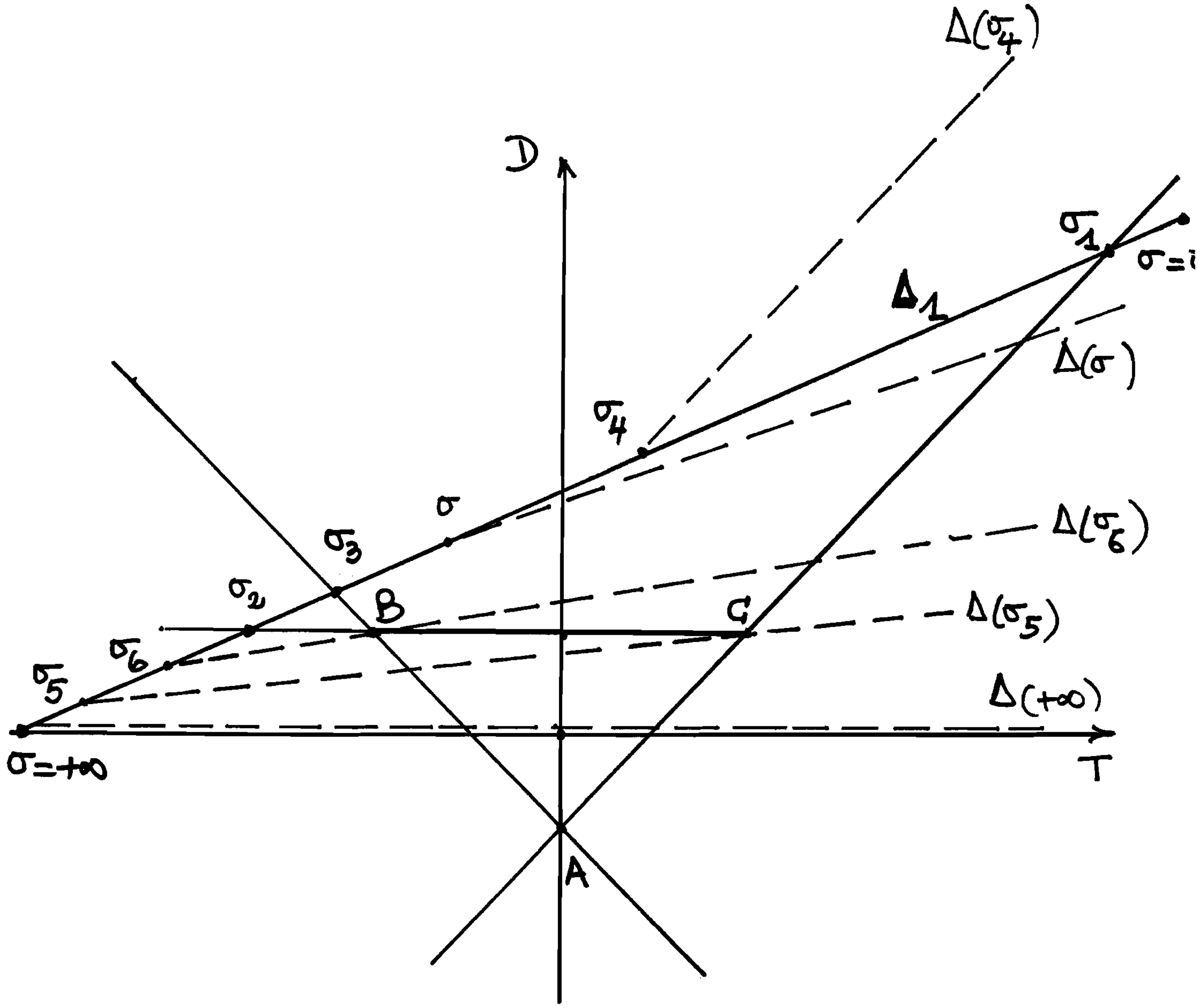
- **GEOMETRICAL METHOD** : LOCATE HALF LINE $\Delta(\sigma)$ IN PLANE

$\Rightarrow$ **QUALITATIVE CONCLUSIONS**

- FOR PLAUSIBLE PROPENSITIES TO CONSUME α (> 0.5), AND FOR SMALL (LABOR) EXTERNALITIES $\varepsilon_{A,\ell}$, LOCAL INDETERMINACY FOR LARGE ELASTICITIES OF CAPITAL LABOR SUBSTITUTION σ (COMPATIBLE WITH COBB DOUGLAS $\sigma = 1$)
- POSSIBILITY OF MULTIPLE STEADY STATES AND TRANSCRITICAL BIFURCATIONS



- SIGNIFICANT BUT NOT TOO LARGE PROPENSITIES TO CONSUME $\alpha(R)$ (>0.5 BUT NOT TOO CLOSE TO 1)
- ELASTICITY OF INTERTEMPORAL SOBSTITUTION $1<\gamma<1/\alpha$. THE CASE $1/\alpha<\gamma$ IS SIMILAR WITH $\sigma=0$ ON LEFT OF LINE (AC)
- SMALL LABOR EXTERNALITY $\varepsilon_{A,e}$



- LARGE PROPENSITIES TO CONSUM ($\alpha(R)$ CLOSE TO 1)
- ELASTICITY OF INTERTEMPORAL SUBSTITUTION $1 < \gamma < 1/\alpha$. THE CASE $1/\alpha < \gamma$ IS SIMILAR WITH $\sigma = 0$ ON THE LEFT OF LINE (AC)
- SMALL LABOR EXTERNALITY $\varepsilon_{A,\ell}$

→ LOCAL STABILITY AND BIFURCUTION ANALYSIS IN 3D (GEOMETRICAL METHODS)

- **CHARACTERISTIC POLYNOMIAL**

$$Q(z) \equiv (z-z_1)(z-z_2)(z-z_3) \equiv z^3 - Tz^2 + Sz - D = 0$$

$T = z_1 + z_2 + z_3$; $S = z_1 z_2 + z_2 z_3 + z_1 z_3$; $D = z_1 z_2 z_3$

- **BIFURCATION LOCI**

$(+1)$: $1 - T + S - D = 0$ $\qquad$ (-1): $1 + T + S + D = 0$

$\begin{pmatrix} z_1 z_2 = 1 \\ |z_1 + z_2| \leqq 2 \end{pmatrix}$: $D = z_3$, $T - D = z_1 + z_2 \Rightarrow$
$S = 1 + D(T-D)$, $|T - D| \leqq 2$

GIVEN D, EXPRESSIONS ARE LINEAR IN (T,S)

$\Rightarrow$ GIVEN D, SIMPLE GRAPHICAL REPRESENTATION IN (T,S) PLANE (TRIANGLE $A_D B_D C_D$)

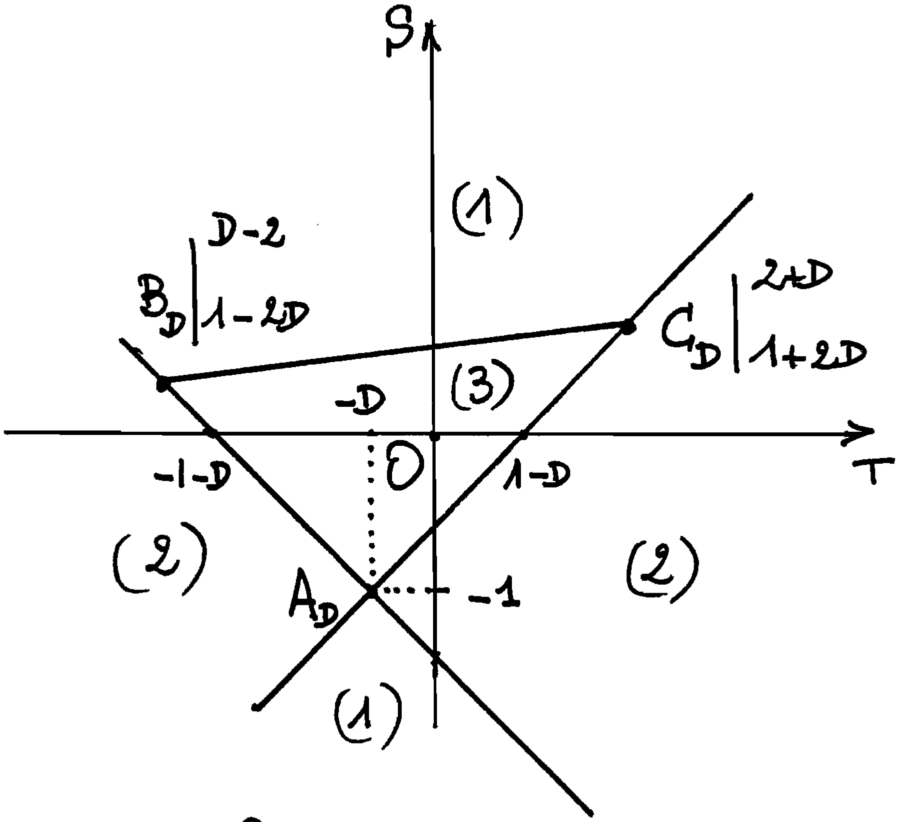
$(+1)$: $S = T - (1-D)$ $\qquad$ LINE $(A_D C_D)$, SLOPE $+1$

(-1): $S = -T - (1+D)$ $\qquad$ LINE $(A_D B_D)$, SLOPE -1

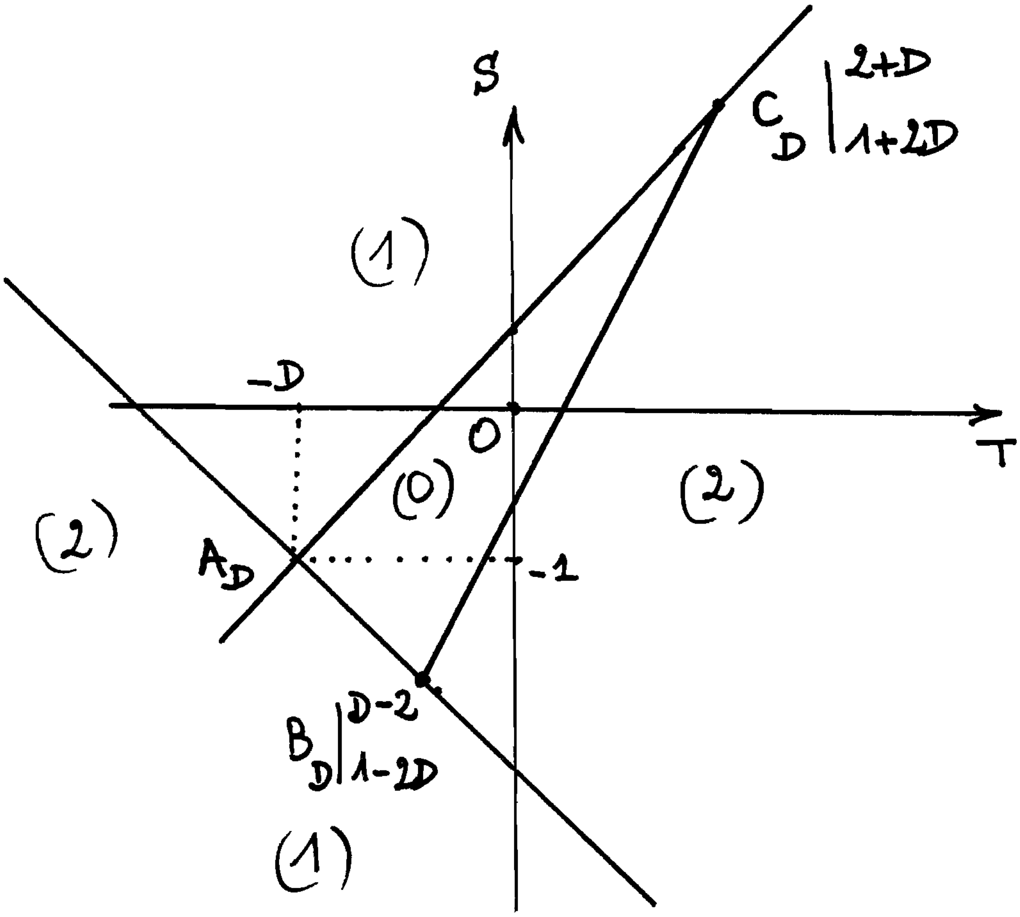
BOTH LINES GO THROUGH $A_D = (-D, -1)$

$\begin{pmatrix} z_1 z_2 = 1 \\ |z_1 + z_2| \leqq 2 \end{pmatrix}$: $\begin{vmatrix} S = DT + 1 - D^2 \\ |T - D| \leqq 2 \end{vmatrix}$ SEGMEN $[B_D C_D]$, SLOPE D

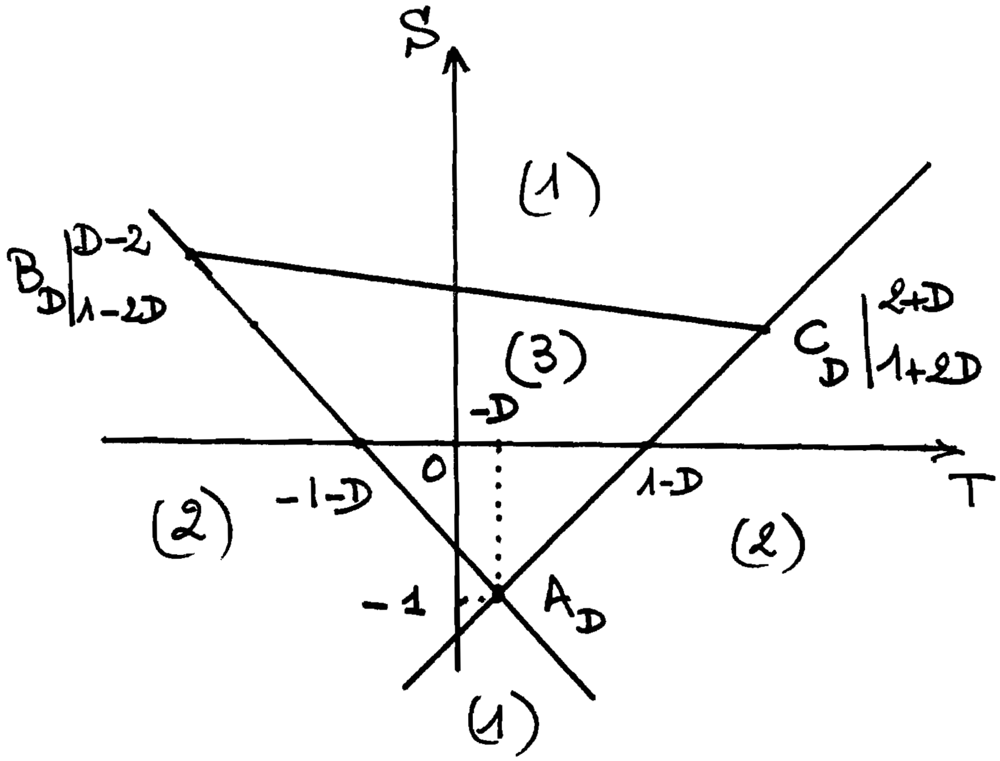
GOES THROUGH $B_D = (D-2, 1-2D)$, $C_D = (D+2, 1+2D)$


S
T
(1)
(3)
(2)
(2)
(1)
B_D | D-2, 1-2D
C_D | 2+D, 1+2D
-D
O
-1-D
1-D
A_D
-1

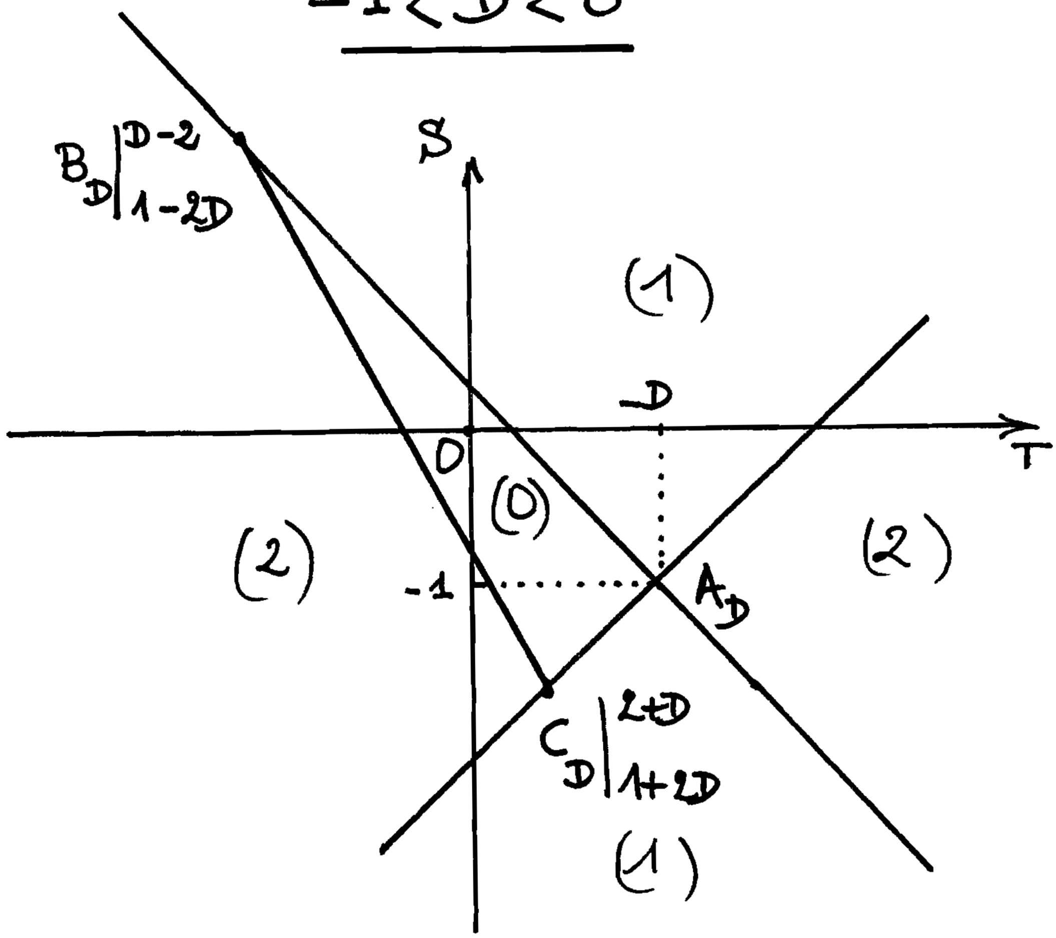
$0 < D < 1$


S
T
C_D | 2+D, 1+2D
(1)
-D
O
(0)
(2)
(2)
A_D
-1
B_D | D-2, 1-2D
(1)

$D > 1$


S
T
(1)
(3)
(2)
(2)
(1)
$B_D \big| \begin{smallmatrix} D-2 \\ 1-2D \end{smallmatrix}$
$C_D \big| \begin{smallmatrix} 2+D \\ 1+2D \end{smallmatrix}$
$-D$
0
$-1-D$
$1-D$
-1
A_D

$-1 < D < 0$


S
T
(1)
(0)
(2)
(2)
(1)
$B_D \big| \begin{smallmatrix} D-2 \\ 1-2D \end{smallmatrix}$
$C_D \big| \begin{smallmatrix} 2+D \\ 1+2D \end{smallmatrix}$
$-D$
0
-1
A_D

$D < -1$