HETEROGENEITY AND AGGREGATION: EQUILIBRIUM "REPRESENTATIVE" AGENTS

- COMPETITIVE EQUILIBRIUM : $\exists$ EQUILIBRIUM REPRESENTATIVE CONSUMER (NEGISHI)

- CENTRAL CONSTRUCT IN MACROECONOMICS: EXPECTED UTILITY MAXIMIZING "REPRESENTATIVE" AGENT MIMICS STOCHASTIC EQUILIBRIUM OVER TIME (ASSET PRICING; BUT ALSO RISK SHARING)

- ABOVE CONSTRUCTION VALID ONLY FOR HOMOGENOUS BELIEFS AND COMPLETE MARKETS. WHAT ABOUT HETEROGENEOUS BELIEFS OR INCOMPLETE MARKETS?

→ COMPLETE ASSET MARKETS

- EXCHANGE ECONOMY ON ARROW-DEBREU SECURITIES $s=1,\dots,S$. PRICE SYSTEM $q \in \mathbb{R}^S_{++}$

- AGENT a CHOOSES PORTFOLIO $y_a=(y_{a1},\dots,y_{aS}) \geq 0$ OF A-D SECURITIES SO AS TO MAXIMIZE

$$E_{\pi_a}[u_a(y_{as})] = \sum_s \pi_{as} u_a(y_{as}) \text{ S.T. } q \cdot y_a = b_a$$

$$\Rightarrow y_a(q, b_a, \pi_a)$$

$\pi_{as} > 0$, $\sum_s \pi_{as} = 1$: INDIVIDUAL BELIEFS

- FOC: $\pi_{as} u_a'(y_{as})/q_s = \lambda_a$

OR $\forall$ ASSET RETURNS R_s (R_0 = RISK FREE)

$$\lambda_a = E_{\pi_a}[R_s u_a'(y_{as})] = R_0 E_{\pi_a}[u_a'(y_{as})]$$

$$= E_{\pi_a}[R_s]\, E_{\pi_a}[u_a'(y_{as})] + \mathrm{COV}_{\pi_a}[R_s, u_a'(y_{as})]$$

→ HOMOGENOUS BELIEFS, $\pi_a = \pi \ \forall a$

- COMPETITIVE EQUILIBRIUM IN EXCHANGE ECONOMY ($b_a = q \cdot \omega_a$): $q^* \gg 0$ SUCH THAT $E_a[y_a^*] = E_a[\omega_a] = \overline{\omega}$, WITH $y_a^* = y_a(q^*, q \cdot \omega_a, \pi)$
- NEGISHI EQUILIBRIUM REPRESENTATIVE AGENT MAXIMIZES $E_\pi[U(y_s)]$ UNDER $q^* \cdot y = q^* \cdot \overline{\omega}$ WITH

$$E_\pi[U(y_s)] = \max_{(y_a)} E_a\left[E_\pi[v_a(y_{as})]\right] \text{ S.T. } E_a[y_{as}] = y_s \ \forall s$$

WHERE $v_a(y) = u_a(y) / E_\pi[u_a'(y_{as}^*)]$ ("NORMALIZED")

- ABOVE ALLOCATION PROBLEM "SEPARATES": $\forall s$,

$$(1) \quad U(y) = \max_{(y_a)} E_a[v_a(y_a)] \quad \text{S.T. } E_a[y_a] = y$$

$\Rightarrow$ IN EQUILIBRIUM, REPRESENTATIVE AGENT CHOOSES $y^* = \overline{\omega}$ AND MIMICS ALL AGENTS' EQUILIBRIUM ASSET PRICING; $\forall$ RETURNS (R_s):

$$(2) \quad E_\pi[R_s U'(\overline{\omega}_s)] = E_\pi[R_s v_a'(y_{as}^*)] = R_0$$

<u>(CCAPM)</u>

$\Rightarrow$ RISK SHARING ("MUTUALIZATION"): IN EQUILIBRIUM, $y_{as}^* = \gamma_a(\overline{\omega}_s)$, WHERE $y_a = \gamma_a(y)$ IS SOLUTION OF ALLOCATION PROBLEM (1)

FOC OF (1) FOR $y = \overline{\omega}_s$: $U'(\overline{\omega}_s) = v_a'(y_{as})$

EQUILIBRIUM FOC : $\pi_s v_a'(y_{as}^*)/q_s^* = R_0$

SO BOTH $v_a'(y_{as})$ AND $v_a'(y_{as}^*)$ ARE INDEPENDENT OF a $\Rightarrow$ $y_{as} = y_{as}^*$ SINCE $E_a[y_{as}] = \overline{\omega}_s = E_a[y_{as}^*]$

$$\Rightarrow \underline{y_{as}^* = \gamma_a(\overline{\omega}_s) \text{ WITH } \gamma_a(y) = (v_a')^{-1}(U'(y))}$$

<u>"MUTUALIZATION"</u>

→ HETEROGENEOUS BELIEFS, π_a DIFFERENT

(CALVET, GRANDMONT AND LEMAIRE, 1999- 2004)

- COMPETITIVE EQUILIBRIUM : $q^* \gg 0$ SUCH THAT

$E_a[y_a^*] = E_a[\omega_a] = \bar{\omega}$ WITH $y_a^* = y_a(q^*, q^* \cdot \omega_a, \pi_a)$

- "AGGREGATE BELIEF" π^0? CONSTRUCT EQUILIBRIUM REPRESENTATIVE AGENT WHO MAXIMIZES $E_{\pi^0}[U(y_s)]$ WITH $U(y)$ GIVEN AS BEFORE BY

(1') $U(y) = \max\limits_{(y_a)} E_a[v_a(y_a)]$ S.T. $E_a[y_a] = y$

WHERE AGAIN $v_a(y) = u_a(y) / E_{\pi_a}[u_a'(y_{as}^*)]$ ("NORMALIZED")

- REPRESENTATIVE AGENT ASSUMED TO MIMIC AGENTS' EQUILIBRIUM ASSET PRICING; $\forall$ RETURNS (R_s):

(2') $E_{\pi^0}[R_s U'(r^0 \bar{\omega}_s)] = E_{\pi_a}[R_s v_a'(y_{as}^*)] = R_0$

AGGREGATE ENDOWMENT MAY NEED TO BE "ADJUSTED" $\bar{\omega} \to r^0 \bar{\omega}$

- FOC(2'): $\forall$ STATE s

$$\frac{\pi_s^0\, U'(r^0\bar{\omega}_s)}{q_s^*} = \frac{\pi_{as}\, v_a'(y_{as}^*)}{q_s^*} = R_0$$

- COMPUTATION OF (π^0, r^0)

– AGGREGATE PROBA $\pi_s^0 = R_0 q_s^* / U'(r^0\bar{\omega}_s)$

WITH $R_0 q_s^* = \pi_s^*$ ("RISK ADJUSTED PROBABILITY")

– ADJUSTMENT COEFFICIENT r^0 SOLVES

$$\sum_s R_0 q_s^* / U'(r^0\bar{\omega}_s) = E_{\pi^*}\left[1/U'(r^0\bar{\omega}_s)\right] = 1$$

PROPOSITION: $\exists$ UNIQUE (π^0, r^0) SATISFYING (2')

"ADJUSTED CCAPM"

(PROOF: $U(y)$ IS STRICTLY CONCAVE,
$U'(y)$ DECREASES FROM $+\infty \searrow$ TO 0
WHEN y $0 \nearrow +\infty$)

→ HETEROGENEOUS BELIEFS : RISK SHARING

- UNDER HOMOGENOUS BELIEFS, $\pi_a = \pi \ \forall a$, EQUILIBRIUM INDIVIDUAL RISKS ARE GIVEN BY $y^*_{as} = \gamma_a(\overline{\omega}_s)$ WHERE $y_a = \gamma_a(y) = (v'_a)^{-1}(U'(y))$ IS SOLUTION OF ALLOCATION PROBLEM DEFINING AGGREGATE UTILITY

$$(1) \quad U(y) = \max_{(y_a)} E_a[v_a(y_a)] \quad \text{S.T. } E_a[y_a] = y$$

WITH $v_a(y) = u_a(y) / E_\pi[u'_a(y^*_{as})]$ ("NORMALIZED")

- UNDER HETEROGENEOUS BELIEFS, EQUILIBRIUM INDIVIDUAL RISKS ARE GIVEN BY $y^*_{as} = \Gamma_a\left(\frac{\pi_{as}}{\pi^0}, r^0 \overline{\omega}_s\right)$ WHERE Γ_a IS INCREASING AND $\Gamma_a(1, y)$ IS SOLUTION $y_a = \gamma_a(y)$ OF SAME ALLOCATION PROBLEM

$$(1') \quad U(y) = \max_{(y_a)} E_a[v_a(y_a)] \ \text{S.T. } E_a[y_a] = y$$

WITH $v_a(y) = u_a(y) / E_{\pi_a}[u'_a(y^*_{as})]$

- FOC OF CONDITIONS DEFINING REPRESENTATIVE AGENT : $\forall$ STATE s

$$\pi_{as}\, v_a'(y_{as}^*) = \pi_s^0\, v_a'(y_{as}^0) = \pi_s^0\, U'(r^0\bar{\omega}_s) \; (= R_0 q_s^*)$$

$$\Rightarrow \left| \begin{array}{l} y_a^* = \Gamma_a\left(\dfrac{\pi_{as}}{\pi_s^0}, r^0\bar{\omega}_s\right) = (v_a')^{-1}\left(\dfrac{\pi_s^0}{\pi_{as}} U'(r^0\bar{\omega}_s)\right) \\ \\ y_{as}^0 = \Gamma_a(1, r^0\bar{\omega}_s) = \gamma_a(r^0\bar{\omega}_s) \end{array} \right.$$

WHERE $\gamma_a(y) = (v_a')^{-1}(U'(y))$ IS SOLUTION OF ALLOCATION PROBLEM $(1')$

$\Rightarrow$ <u>MODIFIED MUTUALIZATION PRINCIPLE</u> <u>WITH DIVERSE BELIEFS</u> (EQUIVALENT CHARACTERIZATION)

- $y \to \gamma_a(y)$: HOMOGENOUS BELIEFS ($\pi_a = \pi^0$) RISK SHARING RULE WITH STANDARD PROPERTIES
- $y_{as}^* - y_{as}^0$: "RESIDUAL RISKS"

→ RESIDUAL RISKS ANALYSIS

$$y^*_{as} - y^o_{as} = g_a(\pi_{as}) - g_a(\pi^o_s)$$

WHERE $g_a(\pi_{as}) = \Gamma_a\left(\frac{\pi_{as}}{\pi^o_s}, r^o \bar{\omega}_s\right)$

$$= (\pi_{as} - \pi^o_s)\, g'_a(\pi^o_s) + \frac{1}{2}(\pi_{as} - \pi^o_s)^2 g''_a(\hat{\pi}_{as})$$

(EXACT 2ND ORDER TAYLOR EXPANSION)

$$g'_a(\pi^o_s) = T_a(y^o_{as}) / \pi^o_s$$

$$g''_a(\pi_s) = T_a(y_{as})(T'_a(y_{as}) - 1) / \pi_s^2$$

WHERE $T_a(y) = -u'_a(y)/u''_a(y)$ IS

<u>ABSOLUTE RISK TOLERANCE</u> ($1/T_a(y)$ IS ABSOLUTE RISK AVERSION)

$g_a(\pi_{as})$ IS CONCAVE IFF $T'_a < 1$

→ MARKET PORTFOLIO ADJUSTMENT COEFFICIENT

$$\frac{1-r^0}{r^0} E_{\pi^0}\left[\rho^0_s\right] = E_a\left[\mathrm{cov}_{\pi^0}\left[\frac{\pi_{as}}{\pi^0_s}, \frac{T^0_{as}}{T^0_s}\right]\right]$$

$$+ \frac{1}{2} E_{\pi^0, a}\left[(\pi_{as} - \pi^0_s)^2 g''_a(\hat{\pi}_{as})\right]$$

(WHERE $T^0_{as} = T_a(y^0_{as})$, $T^0_s = E_a[T^0_{as}]$ is ABSOLUTE RISK TOLERANCE OF REPRESENTATIVE AGENT, $\rho^0_s = r^0 \overline{\omega}_s / T^0_s$ is AGGREGATE RELATIVE RISK AVERSION)

- <u>1ST TERM</u>: EQUAL TO 0 IF NO AGGREGATE RISK ($\overline{\omega}_s = \overline{\omega}_r \Rightarrow y^0_{as} = y^0_{ar}\ \forall a \Rightarrow T^0_{as}, T^0_s$ INDEPENDENT OF STATE s)
- <u>2ND TERM</u>: IS <0 WHEN ALL g_a CONCAVE, OR $T'_a < 1$

⇒ ADJUSTMENT COEFFICIENT $r^0 > 1$ IF

- AGGREGATE RISK IS SMALL
- INDIVIDUAL ABSOLUTE RISK TOLERANCE DOES NOT INCREASE TOO FAST, $T'_a \leqq \eta < 1$
- DISPERSION OF INDIVIDUAL BELIEFS IS SIGNIFICANT

→ SMALL HETEROGENEITY OF BELIEFS, SMALL AGGREGATE RISKS: 2ND ORDER EVALUATIONS

$\bar{\omega}_s - \omega^*$ SMALL $\forall s$ WHERE RISK FREE AGGREGATE INCOME $\omega^* = R_0 q^* \cdot \bar{\omega} = E_{\pi^*}[\bar{\omega}_s]$ WITH $\pi^*_s = R_0 q^*_s$ = RISK ADJUSTED PROBABILITY $\Rightarrow \pi^0_s - \pi^*_s$ ALSO SMALL

- $$\frac{\pi^0_s - \pi^*_s}{\pi^*_s} \simeq \frac{1}{E_a[T^*_a]}(\bar{\omega}_s - \omega^*) + \frac{1}{2}\left(\frac{1 - E_a(\eta^*_a \gamma^*_{a1})}{(E_a[T^*_a])^2}\right)\left((\bar{\omega}_s - \omega^*)^2 - Var_{\pi^*}[\bar{\omega}_s]\right)$$

- $$(r^0 - 1)\frac{\omega^*}{E_a[T^*_a]} \simeq - cov_a\left[cov_{\pi^*}\left[\frac{\pi_{as}}{\pi^*_s}, \bar{\omega}_s\right], \gamma^*_{a2}\right] + \frac{1}{2} E_a\left[Var_{\pi^*}\left[\frac{\pi_{as}}{\pi^*_s}\right]\gamma^*_{a1}(1 - \eta^*_a)\right]$$

WITH NOTATIONS: $T^*_a = T_a(R_0 q^* \cdot \omega_a)$, $\eta^*_a = T'_a(R_0 q^* \cdot \omega_a)$, $\gamma^*_{a1} = T^*_a / E_a[T^*_a]$, $\gamma^*_{a2} = \gamma^*_{a1}(\eta^*_a - E_a[\eta^*_a \gamma^*_{a1}])$

<u>1ST TERM</u> cov_a EQUAL TO 0 WHEN NO AGGREGATE RISKS ($\bar{\omega}_s = \omega^*$), WHEN $\eta^*_a = \eta^*$ (HARA FAMILY) OR WHEN DISTRIBUTIONS OF BELIEFS (π_a) AND OF TASTES + ENDOWMENTS (γ^*_{a2}) INDEPENDENT AMONG AGENTS

<u>2ND TERM</u> POSITIVE ($r^0 > 1$) IF $\eta^*_a < 1 \; \forall a$

- **RISK SHARING RULE** $y^o_{as} = \gamma_a(r^o \bar{\omega}_s) = (v'_a)^{-1}(U'(r^o \bar{\omega}_s))$

$$y^o_{as} \simeq R_o b^o_a + (\bar{\omega}_s - \omega^*)\gamma^*_{a1} + \frac{1}{2}\left((\bar{\omega}_s - \omega^*)^2 - Var_{\pi^*}[\bar{\omega}_s]\right)\gamma^*_{a2}$$

WHERE $b^o_a = q^* \cdot y^o_a$ GIVEN BY

$$R_o(b^o_a - q^* \cdot \omega_a) \simeq \frac{T^*_a(1-\eta^*_a)}{2}\left(Var_{\pi^*}\left[\frac{\pi_{as}}{\pi^*_s}\right] - Var_{\pi^*}\left[\frac{\bar{\omega}_s}{E_a[T^*_a]}\right]\right)$$

UPWARD SHIFT $b^o_a > q^* \cdot \omega_a$ WHEN $\eta^*_a < 1$ IF

π_a NOT TOO CLOSE TO π^*

- **RESIDUAL RISKS**

$$y^*_{as} - y^o_{as} \simeq \left(\frac{\pi_{as} - \pi^o_s}{\pi^o_s}\right)\left(T^*_a + (\bar{\omega}_s - \omega^*)\gamma^*_{a1}\eta^*_a\right)$$
$$+ \frac{1}{2}\left(\frac{\pi_{as} - \pi^o_s}{\pi^o_s}\right)^2 T^*_a(\eta^*_a - 1)$$

→ THE HYPERBOLIC ABSOLUTE RISK TOLERANCE (HARA) FAMILY, $T_a(y) = \theta_a + \eta y$ (GLOBAL RESULTS)

• EXAMPLES

$\eta = 0$: CONSTANT ABSOLUTE RISK AVERSION (CARA)
$u_a(y) = -e^{-y/\theta_a}$, $A_a(y) = 1/T_a(y) = 1/\theta_a$

$\theta_a = 0$: CONSTANT RELATIVE RISK AVERSION, $\rho_a(y) = 1/\eta$

$\eta = 1$: LOGARITHMIC, $u_a(y) = \log(\theta_a + \eta y)$

$\eta > 0$: DECREASING ABSOLUTE RISK AVERSION (DARA)

$\theta_a < 0$: DECREASING RELATIVE RISK AVERSION (DRRA)

• REPRESENTATIVE AGENT BELONGS TO SAME HARA FAMILY WITH $T(y) = \bar{\theta} + \eta y$, $\bar{\theta} = E_a[\theta_a]$

• WHEN HOMOGENOUS PROBABILITIES, $\pi_a = \pi \ \forall a$, OPTIMAL RISK SHARING RULE $y^*_{as} = \gamma_a(\bar{\omega}_s)$ IS LINEAR ($\Leftrightarrow$ TWO FUNDS SEPARATION)

• WITH <u>HETEROGENEOUS BELIEFS</u>, COMMON PROBABILITY RISK SHARING RULE $y^o_{as} = \gamma_a(r^o \bar{\omega}_s)$ AND OBSERVED RISK ALLOCATION $y^*_{as} = \Gamma_a\left(\frac{\pi_{as}}{\pi^o_s}, r^o \bar{\omega}_s\right)$ <u>BOTH LINEAR</u> WITH RESPECT TO AGGREGATE RISKS $\bar{\omega}_s$:

$$\frac{T^o_{as}}{T^o_s} = \frac{\theta_a + \eta y^o_{as}}{\bar{\theta} + \eta r^o \bar{\omega}_s} \text{ IS INDEPENDENT OF STATE } s$$

WHILE OBSERVED DEVIATIONS FROM TWO FUNDS SEPARATION GIVEN BY

$$\frac{\theta_a + \eta y^*_{as}}{\theta_a + \eta y^o_{as}} = \left(\frac{\pi_{as}}{\pi^o_s}\right)^\eta \quad \text{FOR } \eta \neq 0 \text{, AND}$$

$$y^*_{as} - y^o_{as} = \theta_a \log\left(\frac{\pi_{as}}{\pi^o_s}\right) \quad \text{FOR CARA } \eta = 0.$$

$$\Rightarrow \quad \boxed{\begin{array}{lll} r^o > 1 & \text{IF} & \eta < 1 \\ r^o = 1 & \text{IF} & \eta = 1 \\ r^o < 1 & \text{IF} & \eta > 1 \end{array}}$$

"GLOBAL" RESULT

→ DIVERSE BELIEFS AND ASSET PRICING

• ISSUE: GIVEN ASSET WITH RETURNS R_s, DIVERSITY OF BELIEFS $\overset{?}{\Rightarrow}$ "POSITIVE RISK PREMIUM AGGREGATION BIAS" WHERE REPRESENTATIVE AGENT'S RISK PREMIUM EVALUATION

$$E_{\pi^0}[R_s] - R_0 = -\operatorname{cov}_{\pi^0}[R_s, u'(r^0\bar{\omega}_s)] / E_{\pi^0}[u'(r^0\bar{\omega}_s)]$$

IS LOWER THAN ECONOMETRICIAN'S OWN EVALUATION USING "TRUE" PROBABILITY, I.E.

$$E_{\pi}[R_s] - E_{\pi^0}[R_s] > 0 \,?$$

(IF APPLIED TO MARKET PORTFOLIO $R_s = R_s^M = \bar{\omega}_s / q^* \cdot \bar{\omega}$
$\Rightarrow$ "EQUITY PREMIUM")

→ CHANNELS:

1. "DISTORSIONS" OF AGGREGATE PROBABILITY π^0 COMPARED TO "TRUE" PROBABILITY π?
("PESSIMISM", "DOUBT" EFFECTS)

2. ADJUSTMENT COEFFICIENT $r^0 \gtrless 1$?

- RISK PREMIUM AGGREGATION BIAS WITH "NOISY EXPECTATIONS" ("TRUE" PROBABILITY $\bar{\pi} = E_a[\pi_a]$)

$$E_{\bar{\pi}}[R_s] - E_{\pi^0}[R_s] = \mathrm{cov}_{\pi^0}\left[R_s, \frac{\bar{\pi}_s}{\pi^0_s}\right]$$

$$= A + B + C$$

- "PESSIMISM" EFFECT (FIRST TERM)

$$A = -\mathrm{cov}_{\pi^0}\left[R_s, \mathrm{cov}_a\left[\frac{\pi_{as}}{\pi^0_s}, \frac{T^0_{as}}{T^0_s}\right]\right] > 0$$

IF AGENTS WITH LARGER RISK TOLERANCES $T^0_{as} = T_a(y^0_{as})$ THAN AVERAGE $T^0_s = E_a[T^0_{as}]$ ASSIGN LARGER PROBABILITIES π_{as} TO "BAD" STATES INVOLVING LOW RETURNS R_s

- "DOUBT EFFECT" (SECOND TERM)

$$B = \frac{1}{2} \mathrm{cov}_{\pi^0}\left[R_s, E_a\left[\left(\frac{\pi_{as}-\pi_s^0}{\pi_s^0}\right)^2 \frac{T_{as}}{T_s^0}\left(1-T'_{as}\right)\right]\right]$$

IS POSITIVE WHEN ABSOLUTE RISK TOLERANCE DOES NOT INCREASE TOO FAST, $T'_{as} < 1$, IF DISPERSION OF BELIEFS IS SIGNIFICANTLY LARGER FOR "GOOD" STATES INVOLVING LARGE RETURNS R_s

- "ADJUSTMENT EFFECT" (THIRD TERM)

$$C = -\frac{r^0-1}{r^0} \mathrm{cov}_{\pi^0}\left[R_s, \rho_s^0\right]$$

WHERE $\rho_s^0 = \rho(r^0 \bar{\omega}_s)$ IS REPRESENTATIVE AGENTS' RELATIVE RISK AVERSION. IS POSITIVE FOR ASSET WITH R_s POSITIVELY RELATED WITH AGGREGATE CONSUMPTION $\bar{\omega}_s$, IF DECREASING AGGREGATE RISK AVERSION $\rho'(y) < 0$, WHEN $r^0 > 1$.

• SMALL HETEROGENEITY OF BELIEFS, SMALL AGGREGATE RISKS : 2ND ORDER EVALUATION

1. "ADJUSTMENT EFFECT" IS ALWAYS OF 3RD ORDER

IF IN ADDITION DISTRIBUTIONS AMONG AGENTS OF BELIEFS, ENDOWMENTS AND ATTITUDES TOWARD RISK ARE INDEPENDENT:

2. "PESSIMISM EFFECT" IS ALSO OF 3RD ORDER

3. "DOUBT EFFECT"

$$B \simeq \frac{1}{2}\left(1 - E_a[\eta_a^* \gamma_{a1}^*]\right) \mathrm{cov}_{\overline{\pi}}\left[R_s, \mathrm{Var}_a\left[\frac{\pi_{as}}{\overline{\pi}_s}\right]\right]$$

with $\gamma_{a1}^* = T_a^* / E_a[T_a^*]$, IS POSITIVE WHEN $\eta_a^* < 1 \ \forall a$ IF DISPERSION OF BELIEFS $\mathrm{Var}_a[\pi_{as}/\overline{\pi}_s]$ IS LARGER FOR "GOOD" STATES INVOLVING LARGE RETURNS R_s

→ RISK PREMIUM AGGREGATION BIAS IN THE HARA FAMILY ($T_a(y) = \theta_a + \eta y$)

• **LOGARITHMIC UTILITIES** ($\eta = 1$)

$$\frac{\overline{\pi}_s - \pi^0_s}{\pi^0_s} = -\operatorname{cov}_a\left[\frac{\pi_{as}}{\pi^0_s}, \frac{\theta_a + R_0 q^* \cdot \omega_a}{\overline{\theta} + R_0 q^* \cdot \overline{\omega}}\right]$$

RISK PREMIUM AGGREGATION BIAS REDUCES TO "PESSIMISM EFFECT":

$$E_{\overline{\pi}}[R_s] - E_{\pi^0}[R_s] = -\operatorname{cov}_a\left[E_{\pi_a}[R_s], \frac{\theta_a + R_0 q^* \cdot \omega_a}{\overline{\theta} + R_0 q^* \cdot \overline{\omega}}\right]$$

⇒ IS POSITIVE IF AGENTS WITH LARGER RISK TOLERANCE AND/OR INCOME, I.E. WITH LARGER $\theta_a + R_0 q^* \cdot \omega_a$, EXPECT LOWER RETURNS $E_{\pi_a}[R_s]$

⇒ IS 0 FOR EVERY ASSET ($\pi^0 = \overline{\pi}$) IF DISTRIBUTION OF BELIEFS π_a AMONG AGENTS IS INDEPENDENT OF DISTRIBUTIONS OF RISK TOLERANCE θ_a and INCOME $R_0 q^* \cdot \omega_a$

• RISK PREMIUM IN CARA-GAMMA SPECIFICATION

ASSUME:

- $T_a(y) = \theta_a > 0$ $(\eta \simeq 0)$
- CONTINUUM OF STATES $s \in \mathbb{R}_+$, $\overline{\omega}(s) = s$
- BELIEFS $\pi_a(s) \sim$ GAMMA $(\alpha_a > 0, \beta_a > 0)$

(DENSITY $\pi_a(s) = s^{\alpha_a - 1} e^{-s/\beta_a} / (\beta^{\alpha_a} \Gamma(\alpha_a))$

WITH $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, "COMPLETE GAMMA FUNCTION")

- CONTINUUM OF AGENTS, PARAMETERS θ_a, α_a, β_a INDEPENDENTLY DISTRIBUTED IN POPULATION WITH $\log \beta_a \sim \mathcal{N}(m_\beta, v_\beta^2)$

$\Rightarrow$ $\pi^0(s)$ AGGREGATE PROBABILITY HAS (α^0, β^0) GAMMA DISTRIBUTION WITH $\alpha^0 = E_a[\alpha_a]$ AND

$$1/\beta^0 = e^{-m_\beta} e^{(\log \Gamma(\alpha) - E_a[\log \Gamma(\alpha_a)])/\alpha}$$

$$\Rightarrow r^0 - 1 = \overline{\theta} \left(e^{-m_\beta} e^{v_\beta^2/2} - 1/\beta^0 \right)$$

$\Rightarrow$ MEAN $\alpha^0\beta^0$ AND VARIANCE $\alpha^0(\beta^0)^2$ OF AGGREGATE GAMMA DISTRIBUTION $\pi^0(s)$, AS WELL AS r^0, INCREASE AFTER MEAN-PRESERVING SPREAD OF DISTRIBUTION OF α_a ($\log\Gamma(\alpha)$ IS STRICTLY CONVEX FUNCTION)

$\Rightarrow$ $r^0 \nearrow$ IF LARGER VARIANCE OF β_a, $v_\beta^2 \nearrow$

$\Rightarrow$ "PESSIMISM EFFECT" ALWAYS 0 WHEN θ_a AND π_a ARE INDEPENDENTLY DISTRIBUTED IN POPULATION

$\Rightarrow$ "ADJUSTMENT EFFECT" $r^0 \nearrow$ LOWERS RISK PREMIUM AGGREGATION BIAS (AGGREGATE RELATIVE RISK AVERSION INCREASES)

$\Rightarrow$ MARKET PORTFOLIO RISK PREMIUM AGGREGATION BIAS

$$E_{\pi}[\bar{\omega}(s)] - E_{\pi^0}[\bar{\omega}(s)] = E_a\left(E_{\pi_a}(s)\right) - E_{\pi^0}(s) = E_a[\alpha_a\beta_a] - \alpha^0\beta^0$$

$$= \alpha^0\left(e^{m_\beta}\, e^{v_\beta^2/2} - \beta^0\right)$$

DECREASES WITH MEAN-PRESERVING SPREAD OF α_a BUT INCREASES WITH $v_\beta \nearrow$. IS POSITIVE AND LARGE WHEN v_β LARGE (DOMINANT "DOUBT EFFECT")