Nonlinear Mappings in Equilibrium Problems and an Open Problem in Fixed Point Theory

Wataru Takahashi

Keio University, Japan

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $F : C \to C$ is also said to be firmly nonexpansive if $\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$ for all $x, y \in C$. Such two nonlinear mappings are very important in Equilibrium Problems. In 1980, Ray (W. O. Ray, *The fixed point property and unbounded sets in Hilbert space*, Trans. Amer. Math. Soc. **258**, (1980), 531–537) proved the following theorem.

Theorem 1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, the following are equivalent:*

(i) *Every nonexpansive mapping of C into itself has a fixed point in C;*
(ii) *C is bounded.*

Since 1980, Ray's theorem has not been extended to that of a Banach space. We know that a nonexpansive mapping is deduced from a firmly nonexpansive mapping.

Kohsaka and Takahashi (F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166-177) introduced the following nonlinear mapping in a Banach space. Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to C$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$. They proved a fixed point theorem for such mappings. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$.

In this talk, motivated by these operators and results, we try to extend Ray's theorem to that in a Banach space. Then, we solve the open problem. Further, we introduce new nonlinear operators in a Hilbert space and prove some new results for the nonlinear mappings.