Inclusion-Exclusion Families With Respect to Set Functions

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This abstract gives basic properties of inclusion-exclusion families, or interadditive families, with respect to set functions over a nonempty finite set \( X \); for example, the collection of all possible inclusion-exclusion families with respect to set functions over \( X \) is isomorphic to the free bounded distributive lattice generated by \( X \).

Throughout the abstract, \( X \) is assumed to be a nonempty finite set.

1 Families of sets

This section gives a summary of existing results on a lattice structure of families of subsets equipped with a certain partial order (e.g., [1], [2]).

Definition 1. 1. A family \( \mathcal{A} \) of sets is called an antichain if \( \{A,A'\} \subseteq \mathcal{A} \) and \( A \subseteq A' \) together imply \( A = A' \).

2. A family \( \mathcal{H} \) of sets is said to be hereditary if \( H' \subseteq H \in \mathcal{H} \) implies \( H' \in \mathcal{H} \).

Let \( \mathcal{A}(X) \) def \( \{A \subseteq 2^X | A \) is an antichain} \) and \( \mathcal{H}(X) \) def \( \{H \subseteq 2^X | H \) is hereditary\}.

Definition 2. For \( S \subseteq 2^X \), we define \( \text{Max}_S \in \mathcal{A}(X) \) and \( \text{Her}_S \in \mathcal{H}(X) \) by

\[
\text{Max}_S \overset{\text{def}}{=} \{A \mid A \text{ is maximal in } S \text{ with respect to set inclusion } \subseteq\},
\]

\[
\text{Her}_S \overset{\text{def}}{=} \{H \mid H \subseteq S \text{ for some } S \in \mathcal{S}\}.
\]

Definition 3. For \( S, T \subseteq 2^X \).

\[
S \subseteq T \iff S \subseteq \text{Her}_T, \quad S \equiv T \iff \text{Max}_S = \text{Max}_T \iff \text{Her}_S = \text{Her}_T.
\]

Proposition 1. Let \( S, T \subseteq 2^X \).

1. \( S \equiv \text{Max}S \equiv \text{Her}_S \).

2. \( S \equiv T \iff \text{Max}_S = \text{Max}_T \iff \text{Her}_S = \text{Her}_T \).

Obviously, \( \subseteq \) is a preorder on \( 2^{2^X} \), i.e., it is reflexive and transitive, and \( \equiv \) is an equivalence relation on \( 2^{2^X} \). We denote by \([S]\) the equivalence class of \( S \in 2^{2^X} \) with respect to \( \equiv \). Let \( \subseteq \equiv \) be the partial order on the quotient \( 2^{2^X}/\equiv \) induced by \( \subseteq \), i.e.,

\[
[S] \subseteq \equiv [T] \overset{\text{def}}{\iff} S \subseteq T \quad \text{for } S, T \subseteq 2^X.
\]

* This work is partially supported by a grant for the 21st Century COE Program “Creation of Agent-Based Social Systems Sciences” from the Ministry of Education, Culture, Sports, Science and Technology, Japan.
Let $\mathcal{L}(X)$ be the set of lattice polynomials of elements of $X$ defined by

$$
\mathcal{L}(X) \overset{\text{def}}{=} \left\{ \bigvee_{S \in \mathcal{S}} \bigwedge_{x \in S} f \mid S \in 2^{(2^X)} \right\},
$$

where $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$. Then $(\mathcal{L}(X), \land, \lor, 0, 1)$ is the free bounded distributive lattice $(\mathcal{L}(X), \land, \lor, 0, 1)$ generated by $X$, where a bounded lattice is a lattice with the greatest element $1$ and the least element $0$.

**Proposition 2.** Each of $(2^{(2^X)}/\equiv, \subseteq)$, $(A(X), \subseteq)$, $(\mathbb{H}(X), \subseteq)$ is isomorphic to the free bounded distributive lattice $(\mathcal{L}(X), \land, \lor, 0, 1)$ generated by $X$. Especially, $(\mathbb{H}(X), \subseteq)$ is the lattice $(\mathbb{H}(X), \cap, \cup, \emptyset, 2^X)$ of sets. The isomorphism $\varphi: \mathcal{L}(X) \rightarrow 2^{(2^X)}/\equiv$ is given as

$$
\varphi \left( \bigvee_{S \in \mathcal{S}} \bigwedge_{x \in S} f \right) = \{ \{X \setminus S \mid S \in \mathcal{S}\} \}.
$$

2. Set functions and the Choquet integral

The contents of this section are a few modification of existing results (e.g., [3]).

**Definition 4.** A function $\mu: 2^X \rightarrow \mathbb{R}$ is called a set function (with intercept) over $X$. A set function $\mu$ is said to be without intercept if $\mu(\emptyset) = 0$. The essential part of a set function $\mu$ is the set function $\mu_0$ defined by $\mu_0(E) = \mu(E) - \mu(\emptyset)$ for $E \subseteq X$. A set function $\mu$ is said to be modular if $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$ for every pair $E$ and $F$ of subsets of $X$. A modular set function without intercept is said to be additive. Let

$$
\mathbb{SF}(X) \overset{\text{def}}{=} \{ \mu \mid \mu \text{ is a set function over } X \}, \quad \mathbb{SF}_0(X) \overset{\text{def}}{=} \{ \mu \in \mathbb{SF}(X) \mid \mu(\emptyset) = 0 \}.
$$

Hereinafter $\mu$ is assumed to be a set function over $X$, i.e., $\mu \in \mathbb{SF}(X)$.

**Definition 5.** (cf. [4]) The Choquet integral $$(C) \int f(x) \, d\mu(x)$$ of a function $f: X \rightarrow \mathbb{R}$ with respect to $\mu$ is defined by

$$(C) \int f \, d\mu \overset{\text{def}}{=} \mu(\emptyset) + \sum_{i=1}^{\lvert X \rvert} [f(x_i) - f(x_{i-1})] [\mu(A_i) - \mu(\emptyset)],$$

where $x_1, x_2, \ldots, x_{\lvert X \rvert}$ is a permutation of the elements of $X$ satisfying the condition $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{\lvert X \rvert})$, $f(x_0) \overset{\text{def}}{=} 0$, and $A_i \overset{\text{def}}{=} \{x_i, x_{i+1}, \ldots, x_{\lvert X \rvert}\}$ $(i = 1, 2, \ldots, \lvert X \rvert)$.

Obviously it holds that

$$(C) \int f \, d\mu = \mu(\emptyset) + (C) \int f \, d\mu_0.$$

There are two distinct definitions of the Choquet integral over $E \subseteq X$:

$$(C) \int_E f \, d\mu \overset{\text{def}}{=} (C) \int (f \{E\}) \, d(\mu \mid 2^E), \quad (C) \int_E f \, d\mu \overset{\text{def}}{=} (C) \int f \cdot 1_E \, d\mu,$$

where $1_E$ is the indicator of $E$. In this abstract, however, we may adopt whichever one.
Definition 6. The Möbius transform $\mu^M$ of $\mu$ is a set function over $X$ defined by

$$\mu^M(E) \overset{\text{def}}{=} \sum_{F \subseteq E} (-1)^{|E \setminus F|} \mu(F).$$

By definition, $\mu^M(\emptyset) = \mu(\emptyset)$. In addition, $(\mu_0)^M(E) = \mu^M(E)$ for every $E \in 2^X \setminus \{\emptyset\}$.

Definition 7. A subset $F$ of $X$ is called a focus, or a focal element, of $\mu$ if $\mu^M(F) \neq 0$. The family of foci of $\mu$ is denoted by $\mathcal{F}(\mu)$; that is, $\mathcal{F}(\mu) \overset{\text{def}}{=} \{ F \subseteq X \mid \mu^M(F) \neq 0 \}$.

Obviously $\mathcal{F}(\mu_0) = \mathcal{F}(\mu) \setminus \{\emptyset\}$.

Definition 8. $\mu$ is said to be $k$-modular if $k = \max\{|F| \mid F \in \mathcal{F}(\mu)\}$, where $\max\emptyset \overset{\text{def}}{=} 0$. A $k$-modular set function without intercept is said to be $k$-additive.

The following proposition includes the definition of null set.

Proposition 3. Let $N \subseteq X$. The following conditions are equivalent to each other.

(a) $N$ is a null set with respect to $\mu$.
(b) $\mu(E \setminus N) = \mu(E)$ whenever $E \subseteq X$.
(c) $N \subseteq X \setminus \bigcup \mathcal{F}(\mu)$.
(d) For every $f : X \to \mathbb{R}$,

$$(C) \int_X f \, d\mu = (C) \int_{X\setminus N} f \, d\mu.$$  

$X \setminus \bigcup \mathcal{F}(\mu)$ is the greatest null set. If $N$ is a null set, then $\mu(N) = \mu(\emptyset)$. The family of null sets with respect to $\mu_0$ coincides with the family of null sets with respect to $\mu$.

3 Inclusion-exclusion families

The contents of this section are a few modification of existing results (e.g., [3]). The following theorem includes the definition of inclusion-exclusion family.

Theorem 1. Let $\mu$ be a set function over $X$ and $S$ a family of subsets of $X$. The following conditions are equivalent to each other.

(a) $S$ is an inclusion-exclusion family, or an interadditive family, with respect to $\mu$.
(b) For every $E \subseteq X$

$$\mu(E) = \sum_{T \subseteq S, T \neq \emptyset} (-1)^{|T|+1} \mu\left(\bigcap T \cap E\right). \quad (2)$$

(c) Eq. (2) holds for every $E \in 2^X \setminus \text{Her} S$.
(d) $\mathcal{F}(\mu) \subseteq S$, or equivalently $\mu^M(E) = 0$ for every $E \in 2^X \setminus \text{Her} S$.
(e) There exists a collection $\{\mu_S\}_{S \subseteq S}$ of set functions, each $\mu_S$ of which is defined on $2^S$, such that for every $E \subseteq X$

$$\mu(E) = \sum_{S \subseteq S} \mu_S(E \cap S).$$

(f) There exists a collection $\{\mu_S\}_{S \subseteq S}$ of set functions, each $\mu_S$ of which is defined on $2^S$, such that for every function $f : X \to \mathbb{R}$

$$(C) \int_X f \, d\mu = \sum_{S \subseteq S} (C) \int_S f \, d\mu_S.$$  

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By the theorem above, \( \mathcal{F}(\mu) \) itself is an inclusion-exclusion family and one of the least ones with respect to \( \subseteq \). Hence \( \text{Max}\mathcal{F}(\mu) \) is the least antichain inclusion-exclusion family and \( \text{Her}\mathcal{F}(\mu) \) is the least hereditary inclusion-exclusion family. A family \( S \) of subsets of \( X \) is an inclusion-exclusion family with respect to \( \mu \) if \( S \cup \{\emptyset\} \) is an inclusion-exclusion family with respect to \( \mu \).

**Proposition 4.** Let \( S \) be an inclusion-exclusion family.

1. \( X \setminus \bigcup S \) is a null set.
2. If \( N \) is a null set, then \( \{S \setminus N \mid S \in S\} \) also is an inclusion-exclusion family.

**Proposition 5.** Let \( k \) be a nonnegative integer less than or equal to \( |X| \). \( \mu \) is at most \( k \)-modular iff \( \binom{|X|}{k} \) is an inclusion-exclusion family.

If we consider only set functions without intercept, since there is no \( \mu \in \mathcal{SF}_0(X) \) such that \( \mathcal{F}(\mu) = \{\emptyset\} \), we may exclude \( \emptyset \) from consideration. Then the collection of all possible inclusion-exclusion families with respect to set functions without intercept over \( X \) is isomorphic to the free upper-bounded distributive lattice generated by \( X \), where an upper-bounded lattice is a lattice with the greatest element

1. Since \( \emptyset \notin \mathcal{F}(\mu) \) for all \( \mu \in \mathcal{SF}_0(X) \) and \( S \setminus \{\emptyset\} \equiv S \) for all \( S \subseteq 2^X \) except \( S = \{\emptyset\} \), instead of excluding \( \emptyset \) from the collection of families of subsets, we can exclude \( \emptyset \) from families of subsets. Let \( A_{\emptyset}(X) \equiv \{A \setminus \{\emptyset\} \mid A \in \mathcal{A}(X)\} \) and \( H_{\emptyset}(X) \equiv \{\mathcal{H}\setminus\{\emptyset\} \mid \mathcal{H} \in \mathcal{H}(X)\} \). Note that \( 2(2^X\setminus\{\emptyset\}) = \{\mathcal{F}(\mu) \mid \mu \in \mathcal{SF}_0(X)\} \), \( A_{\emptyset}(X) = \{\text{Max}\mathcal{F}(\mu) \mid \mu \in \mathcal{SF}_0(X)\} = \mathcal{A}(X) \setminus \{\emptyset\} \), and \( H_{\emptyset}(X) = \{(\text{Her}\mathcal{F}(\mu)) \setminus \{\emptyset\} \mid \mu \in \mathcal{SF}_0(X)\} \). In addition, a set function without intercept with an inclusion-exclusion family \( S \) is determined by its values on \( \langle \text{Her}S \rangle \setminus \{\emptyset\} \).

**Proposition 6.** Each of \( (2^{2^X} \setminus \{\emptyset\})/\equiv, \subseteq \equiv, \subseteq \equiv, (A_{\emptyset}(X), \subseteq), (H_{\emptyset}(X) \setminus \{\emptyset\}, \subseteq), (H_{\emptyset}(X), \subseteq) \) is isomorphic to the free upper-bounded distributive lattice \( \mathcal{L}(X) \setminus \{\emptyset\}, \wedge, \vee, 1 \) generated by \( X \). Especially, \( (H_{\emptyset}(X), \subseteq) \) is the lattice \( (H_{\emptyset}(X), \cup, \cap, 2^X \setminus \{\emptyset\}) \) of sets. The isomorphism \( \varphi : \mathcal{L}(X) \setminus \{\emptyset\} \to (2^{2^X} \setminus \{\emptyset\})/\equiv \) is given by Eq. (1) provided that \( \{\emptyset\} \) is identified with \( \emptyset \).

**References**