Inclusion-Exclusion Families With Respect to Set Functions*

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This abstract gives basic properties of inclusion-exclusion families, or interadditive families, with respect to set functions over a nonempty finite set X; for example, the collection of all possible inclusion-exclusion families with respect to set functions over X is isomorphic to the free bounded distributive lattice generated by X.

Throughout the abstract, X is assumed to be a nonempty finite set.

1 Families of sets

This section gives a summary of existing results on a lattice structure of families of subsets equipped with a certain partial order (e.g., [1], [2]).

Definition 1. *1. A family $\mathcal{A}$ of sets is called an* antichain *if $\{A, A'\} \subseteq \mathcal{A}$ and $A \subseteq A'$ together imply $A = A'$.*

2. A family $\mathcal{H}$ of sets is said to be hereditary *if $H' \subseteq H \in \mathcal{H}$ implies $H' \in \mathcal{H}$.*

Let $\mathbb{A}(X) \stackrel{\text{def}}{=} \{\mathcal{A} \subseteq 2^X \mid \mathcal{A}$ is an antichain$\}$ and $\mathbb{H}(X) \stackrel{\text{def}}{=} \{\mathcal{H} \subseteq 2^X \mid \mathcal{H}$ is hereditary$\}$.

Definition 2. *For $\mathcal{S} \subseteq 2^X$, we define $\mathrm{Max}\mathcal{S} \in \mathbb{A}(X)$ and $\mathrm{Her}\mathcal{S} \in \mathbb{H}(X)$ by*

$$\mathrm{Max}\mathcal{S} \stackrel{\text{def}}{=} \{A \mid A \text{ is maximal in } \mathcal{S} \text{ with respect to set inclusion } \subseteq\},$$
$$\mathrm{Her}\mathcal{S} \stackrel{\text{def}}{=} \{H \mid H \subseteq S \text{ for some } S \in \mathcal{S}\}.$$

Definition 3. *For $\mathcal{S}$, $\mathcal{T} \subseteq 2^X$.*

$$\mathcal{S} \sqsubseteq \mathcal{T} \stackrel{\text{def}}{\iff} \mathcal{S} \subseteq \mathrm{Her}\mathcal{T}, \qquad \mathcal{S} \equiv \mathcal{T} \stackrel{\text{def}}{\iff} \mathcal{S} \sqsubseteq \mathcal{T} \text{ and } \mathcal{T} \sqsubseteq \mathcal{S}.$$

Proposition 1. *Let $\mathcal{S}$, $\mathcal{T} \subseteq 2^X$.*

1. $\mathcal{S} \equiv \mathrm{Max}\mathcal{S} \equiv \mathrm{Her}\mathcal{S}$.
2. $\mathcal{S} \equiv \mathcal{T} \iff \mathrm{Max}\mathcal{S} = \mathrm{Max}\mathcal{T} \iff \mathrm{Her}\mathcal{S} = \mathrm{Her}\mathcal{T}$.

Obviously, $\sqsubseteq$ is a preorder on $2^{(2^X)}$, i.e., it is reflexive and transitive, and $\equiv$ is an equivalence relation on $2^{(2^X)}$. We denote by $[\mathcal{S}]$ the equivalence class of $\mathcal{S} \in 2^{(2^X)}$ with respect to $\equiv$. Let $\sqsubseteq_\equiv$ be the partial order on the quotient $2^{(2^X)}/\equiv$ induced by $\sqsubseteq$, i.e.,

$$[\mathcal{S}] \sqsubseteq_\equiv [\mathcal{T}] \stackrel{\text{def}}{\iff} \mathcal{S} \sqsubseteq \mathcal{T} \quad \text{for } \mathcal{S},\ \mathcal{T} \subseteq 2^X.$$

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Let $\mathfrak{L}(X)$ be the set of lattice polynomials of elements of X defined by

$$\mathfrak{L}(X) \stackrel{\text{def}}{=} \left\{ \bigvee_{S \in \mathcal{S}} \bigwedge_{x \in S} x \;\middle|\; \mathcal{S} \in 2^{(2^X)} \right\},$$

where $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$. Then $(\mathfrak{L}(X), \wedge, \vee, 0, 1)$ is the free bounded distributive lattice $(\mathfrak{L}(X), \wedge, \vee, 0, 1)$ generated by X, where a bounded lattice is a lattice with the greatest element 1 and the least element 0.

Proposition 2. *Each of* $(2^{(2^X)}/\equiv, \sqsubseteq_{\equiv})$, $(\mathbb{A}(X), \sqsubseteq)$, $(\mathbb{H}(X), \sqsubseteq)$ *is isomorphic to the free bounded distributive lattice* $(\mathfrak{L}(X), \wedge, \vee, 0, 1)$ *generated by* X*. Especially,* $(\mathbb{H}(X), \sqsubseteq)$ *is the lattice* $(\mathbb{H}(X), \cap, \cup, \emptyset, 2^X)$ *of sets. The isomorphism* $\varphi : \mathfrak{L}(X) \to 2^{(2^X)}/\equiv$ *is given as*

$$\varphi\left(\bigvee_{S \in \mathcal{S}} \bigwedge_{x \in S} x \right) = [\{X \setminus S \mid S \in \mathcal{S}\}]. \tag{1}$$

2 Set functions and the Choquet integral

The contents of this section are a few modification of existing results (e.g., [3]).

Definition 4. *A function* $\mu : 2^X \to \mathbb{R}$ *is called a* set function *(*with intercept*) over* X*. A set function* μ *is said to be* without intercept *if* $\mu(\emptyset) = 0$*. The* essential part *of a set function* μ *is the set function* $\mu_\emptyset$ *defined by* $\mu_\emptyset(E) = \mu(E) - \mu(\emptyset)$ *for* $E \subseteq X$*. A set function* μ *is said to be* modular *if* $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$ *for every pair* E *and* F *of subsets of* X*. A modular set function without intercept is said to be* additive*. Let*

$$\mathbb{SF}(X) \stackrel{\text{def}}{=} \{\mu \mid \mu \textit{ is a set function over } X\}, \qquad \mathbb{SF}_\emptyset(X) \stackrel{\text{def}}{=} \{\mu \in \mathbb{SF}(X) \mid \mu(\emptyset) = 0\}.$$

Hereinafter μ is assumed to be a set function over X, i.e., $\mu \in \mathbb{SF}(X)$.

Definition 5. *(cf. [4]) The* Choquet integral $(\mathrm{C}) \int f(x)\, d\mu(x)$ *of a function* $f : X \to \mathbb{R}$ *with respect to* μ *is defined by*

$$(\mathrm{C}) \int f\, d\mu \stackrel{\text{def}}{=} \mu(\emptyset) + \sum_{i=1}^{|X|} [f(x_i) - f(x_{i-1})]\, [\mu(A_i) - \mu(\emptyset)],$$

where $x_1, x_2, \ldots, x_{|X|}$ *is a permutation of the elements of* X *satisfying the condition* $f(x_1) \le f(x_2) \le \cdots \le f(x_{|X|})$*,* $f(x_0) \stackrel{\text{def}}{=} 0$*, and* $A_i \stackrel{\text{def}}{=} \{x_i, x_{i+1}, \ldots, x_{|X|}\}$ *(*$i = 1, 2, \ldots, |X|$*).*

Obviously it holds that

$$(\mathrm{C}) \int f\, d\mu = \mu(\emptyset) + (\mathrm{C}) \int f\, d\mu_\emptyset.$$

There are two distinct definitions of the Choquet integral over $E \subseteq X$:

$$(\mathrm{C}) \int_E f\, d\mu \stackrel{\text{def}}{=} (\mathrm{C}) \int (f \upharpoonright E)\, d(\mu \upharpoonright 2^E), \qquad (\mathrm{C}) \int_E f\, d\mu \stackrel{\text{def}}{=} (\mathrm{C}) \int f \cdot 1_E\, d\mu,$$

where 1_E is the indicator of E. In this abstract, however, we may adopt whichever one.

Definition 6. *The* Möbius transform μ^{M} *of μ is a set function over X defined by*

$$\mu^{\mathrm{M}}(E) \stackrel{\text{def}}{=} \sum_{F\subseteq E}(-1)^{|E\setminus F|}\mu(F).$$

By definition, $\mu^{\mathrm{M}}(\emptyset)=\mu(\emptyset)$. In addition, $(\mu_{\emptyset})^{\mathrm{M}}(E)=\mu^{\mathrm{M}}(E)$ for every $E\in 2^X\setminus\{\emptyset\}$.

Definition 7. *A subset F of X is called a* focus, *or a* focal element, *of μ if $\mu^{\mathrm{M}}(F)\neq 0$. The family of foci of μ is denoted by $\mathcal{F}(\mu)$; that is, $\mathcal{F}(\mu) \stackrel{\text{def}}{=} \{F\subseteq X \mid \mu^{\mathrm{M}}(F)\neq 0\}$.*

Obviously $\mathcal{F}(\mu_{\emptyset})=\mathcal{F}(\mu)\setminus\{\emptyset\}$.

Definition 8. *μ is said to be k-*modular *if $k=\max\{|F| \mid F\in\mathcal{F}(\mu)\}$, where $\max\emptyset \stackrel{\text{def}}{=} 0$. A k-modular set function without intercept is said to be k-*additive.

The following proposition includes the definition of null set.

Proposition 3. *Let $N\subseteq X$. The following conditions are equivalent to each other.*

(a) *N is a* null set *with respect to μ.*
(b) *$\mu(E\setminus N)=\mu(E)$ whenever $E\subseteq X$.*
(c) *$N\subseteq X\setminus\bigcup\mathcal{F}(\mu)$.*
(d) *For every $f:X\to\mathbb{R}$,*

$$(\mathrm{C})\int_X f\,d\mu \;=\; (\mathrm{C})\int_{X\setminus N} f\,d\mu.$$

$X\setminus\bigcup\mathcal{F}(\mu)$ is the greatest null set. If N is a null set, then $\mu(N)=\mu(\emptyset)$. The family of null sets with respect to $\mu_{\emptyset}$ coincides with the family of null sets with respect to μ.

3 Inclusion-exclusion families

The contents of this section are a few modification of existing results (e.g., [3]). The following theorem includes the definition of inclusion-exclusion family.

Theorem 1. *Let μ be a set function over X and $\mathcal{S}$ a family of subsets of X. The following conditions are equivalent to each other.*

(a) *$\mathcal{S}$ is an* inclusion-exclusion family, *or an* interadditive family, *with respect to μ.*
(b) *For every $E\subseteq X$*

$$\mu(E) \;=\; \sum_{\mathcal{T}\subseteq\mathcal{S},\ \mathcal{T}\neq\emptyset}(-1)^{|\mathcal{T}|+1}\mu\Big(\bigcap\mathcal{T}\cap E\Big). \tag{2}$$

(c) *Eq.* (2) *holds for every $E\in 2^X\setminus\mathrm{Her}\mathcal{S}$.*
(d) *$\mathcal{F}(\mu)\sqsubseteq\mathcal{S}$, or equivalently $\mu^{\mathrm{M}}(E)=0$ for every $E\in 2^X\setminus\mathrm{Her}\mathcal{S}$.*
(e) *There exists a collection $\{\mu_S\}_{S\in\mathcal{S}}$ of set functions, each μ_S of which is defined on 2^S, such that for every $E\subseteq X$*

$$\mu(E) \;=\; \sum_{S\in\mathcal{S}}\mu_S(E\cap S).$$

(f) *There exists a collection $\{\mu_S\}_{S\in\mathcal{S}}$ of set functions, each μ_S of which is defined on 2^S, such that for every function $f:X\to\mathbb{R}$*

$$(\mathrm{C})\int_X f\,d\mu \;=\; \sum_{S\in\mathcal{S}}(\mathrm{C})\int_S f\,d\mu_S.$$

By the theorem above, $\mathcal{F}(\mu)$ itself is an inclusion-exclusion family and one of the least ones with respect to $\sqsubseteq$. Hence $\mathrm{Max}\mathcal{F}(\mu)$ is the least antichain inclusion-exclusion family and $\mathrm{Her}\mathcal{F}(\mu)$ is the least hereditary inclusion-exclusion family. A family $\mathcal{S}$ of subsets of X is an inclusion-exclusion family with respect to $\mu_\emptyset$ iff $\mathcal{S} \cup \{\emptyset\}$ is an inclusion-exclusion family with respect to μ.

Proposition 4. *Let $\mathcal{S}$ be an inclusion-exclusion family.*

1. *$X \setminus \bigcup \mathcal{S}$ is a null set.*
2. *If N is a null set, then $\{S \setminus N \mid S \in \mathcal{S}\}$ also is an inclusion-exclusion family.*

Proposition 5. *Let k be a nonnegative integer less than or equal to $|X|$. μ is at most k-modular iff $\binom{X}{k}$ is an inclusion-exclusion family.*

If we consider only set functions without intercept, since there is no $\mu \in \mathbb{SF}_\emptyset(X)$ such that $\mathcal{F}(\mu) = \{\emptyset\}$, we may exclude $\{\emptyset\}$ from consideration. Then the collection of all possible inclusion-exclusion families with respect to set functions without intercept over X is isomorphic to the free upper-bounded distributive lattice generated by X, where an upper-bounded lattice is a lattice with the greatest element 1. Since $\emptyset \notin \mathcal{F}(\mu)$ for all $\mu \in \mathbb{SF}_\emptyset(X)$ and $\mathcal{S} \setminus \{\emptyset\} \equiv \mathcal{S}$ for all $\mathcal{S} \subseteq 2^X$ except $\mathcal{S} = \{\emptyset\}$, instead of excluding $\{\emptyset\}$ from the collection of families of subsets, we can exclude $\emptyset$ from families of subsets. Let $\mathbb{A}_{\setminus\emptyset}(X) \stackrel{\mathrm{def}}{=} \{\mathcal{A} \setminus \{\emptyset\} \mid \mathcal{A} \in \mathbb{A}(X)\}$ and $\mathbb{H}_{\setminus\emptyset}(X) \stackrel{\mathrm{def}}{=} \{\mathcal{H} \setminus \{\emptyset\} \mid \mathcal{H} \in \mathbb{H}(X)\}$. Note that $2^{(2^X \setminus \{\emptyset\})} = \{\mathcal{F}(\mu) \mid \mu \in \mathbb{SF}_\emptyset(X)\}$, $\mathbb{A}_{\setminus\emptyset}(X) = \{\mathrm{Max}\mathcal{F}(\mu) \mid \mu \in \mathbb{SF}_\emptyset(X)\} = \mathbb{A}(X) \setminus \{\{\emptyset\}\}$, and $\mathbb{H}_{\setminus\emptyset}(X) = \{(\mathrm{Her}\mathcal{F}(\mu)) \setminus \{\emptyset\} \mid \mu \in \mathbb{SF}_\emptyset(X)\}$. In addition, a set function without intercept with an inclusion-exclusion family $\mathcal{S}$ is determined by its values on $(\mathrm{Her}\mathcal{S}) \setminus \{\emptyset\}$.

Proposition 6. *Each of $((2^{(2^X)} \setminus \{\{\emptyset\}\})/\equiv, \sqsubseteq_\equiv)$, $(2^{(2^X \setminus \{\emptyset\})}/\equiv, \sqsubseteq_\equiv)$, $(\mathbb{A}_{\setminus\emptyset}(X), \sqsubseteq)$, $(\mathbb{H}(X) \setminus \{\{\emptyset\}\}, \sqsubseteq)$, $(\mathbb{H}_{\setminus\emptyset}(X), \sqsubseteq)$ is isomorphic to the free upper-bounded distributive lattice $(\mathfrak{L}(X) \setminus \{0\}, \wedge, \vee, 1)$ generated by X. Especially, $(\mathbb{H}_{\setminus\emptyset}(X), \sqsubseteq)$ is the lattice $(\mathbb{H}_{\setminus\emptyset}(X), \cup, \cap, 2^X \setminus \{\emptyset\})$ of sets. The isomorphism $\varphi : \mathfrak{L}(X) \setminus \{0\} \to (2^{(2^X)} \setminus \{\{\emptyset\}\})/\equiv$ is given by Eq.* (1) *provided that $[\{\emptyset\}]$ is identified with $[\emptyset]$.*

References

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