REGULARITY OF SOLUTION MAPPINGS FOR INEQUALITY SYSTEMS

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Metric regularity is one of the central concepts in variational analysis.

Definition. A set-valued mapping $F\colon X \rightrightarrows Y$ is said to be *metrically regular* at $(\bar{x}, \bar{y})$ if there exists $\kappa \geq 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x))$$

for (x, y) close to $(\bar{x}, \bar{y})$. The infimum of such κ is called *the modulus of metric regularity* at $(\bar{x}, \bar{y})$ and denoted by $\operatorname{reg} F(\bar{x}|\bar{y})$.

It can represent regularity of solution mappings for inequality systems, which naturally appears in optimization problems. First, using a problem of financial games, we illustrate roles of metric regularity.

In *one way trading game*, the trader will exchange some amount of dollars into as much yen as possible in a certain period. We recently formulated the problem as a variational problem in $L^1[0,1]$;

$$\begin{aligned} \text{minimize} & \int_m^M \frac{p}{\int_m^p r x(r) dr + m(1 - \int_m^p x(r) dr)} f(p) dp \\ \text{subject to} & \int_m^M x(r) dr = 1,\ x(r) \geq 0,\ x \in L^1[m, M] \end{aligned}$$

We can show that the solution mapping for the constraint system is metrically regular. Then an optimal solution is obtained under suitable assumptions (Fujiwara-Iwama-Sekiguchi, COCOON2008);

$$\bar{x}(r) = \begin{cases} \frac{m}{(\alpha - m)\sqrt{\alpha f(\alpha)}} \left\{ \frac{3r - m}{2(r - m)} \sqrt{\frac{f(r)}{r}} + \frac{f'(r)}{2} \sqrt{\frac{r}{f(r)}} \right\},\ r \in [\alpha, \beta]; \\ 0,\ r \notin [a, \beta], \end{cases}$$

where the constants α, β satisfy $\beta(\beta - m) f(\beta) = \int_\beta^M r f(r) dr$, $\int_\alpha^\beta \bar{x}(r) dr = 1$.

Second, we discuss theoretical properties of metric regularity. The following is the classical theorem on matrices;

$$\inf_{B \in \mathbb{R}^{n \times n}} \{ \|B\| \mid A + B \text{ is singular } \} = \frac{1}{\|A\|}$$

This formula enables us to measure a robustness of regularity of matrix equalities. It was extended to general set-valued mappings between Euclidean spaces using metric regularity. However it is known that the formula does not holds in Banach spaces generally. We prove a version of the theorem in continuous function spaces. Let Ω be a compact Hausdorff space and $X = C(\Omega)^n, Y = C(\Omega)^m$. Define a set-valued mapping $F\colon X \rightrightarrows Y$ by

$$F(x) \ni y \;\Leftrightarrow\; f(x(\omega)) \leq y(\omega),$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable, and the order is coordinatewise. Metric regularity of F represents stability of the solution mapping for the inequality system.

Theorem (Ioffe-Sekiguchi, Math. Program. 2009)**.** *For a point $(\bar{x}, \bar{y})$ in the graph of F,*

$$\inf_{g\colon X\to Y}\{\operatorname{Lip} g(\bar{x}) \mid F + g \textit{ is not metrically regular at } (\bar{x}, \bar{y} + g(\bar{x}))\} = \frac{1}{\operatorname{reg} F(\bar{x}|\bar{y})},$$

where $\operatorname{Lip} g$ *is the Lipschitz constant of* g *at* $\bar{x}$*.*

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