REGULARITY OF SOLUTION MAPPINGS FOR INEQUALITY SYSTEMS

YOSHIYUKI SEKIGUCHI

Metric regularity is one of the central concepts in variational analysis.

**Definition.** A set-valued mapping $F: X \rightrightarrows Y$ is said to be **metrically regular** at $(\bar{x}, \bar{y})$ if there exists $\kappa \geq 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x))$$

for $(x, y)$ close to $(\bar{x}, \bar{y})$. The infimum of such $\kappa$ is called the **modulus of metric regularity** at $(\bar{x}, \bar{y})$ and denoted by $\text{reg} F(\bar{x}|\bar{y})$.

It can represent regularity of solution mappings for inequality systems, which naturally appears in optimization problems. First, using a problem of financial games, we illustrate roles of metric regularity.

In one way trading game, the trader will exchange some amount of dollars into as much yen as possible in a certain period. We recently formulated the problem as a variational problem in $L^1[0,1]$:

$$\begin{align*}
\text{minimize} & \quad \int_{m}^{M} \int_{p}^{r} x(r)dr + m(1 - \int_{m}^{p} x(r)dr) f(p)dp \\
\text{subject to} & \quad \int_{m}^{M} x(r)dr = 1, \quad x(r) \geq 0, \quad x \in L^1[m, M]
\end{align*}$$

We can show that the solution mapping for the constraint system is metrically regular. Then an optimal solution is obtained under suitable assumptions (Fujiwara-Iwama-Sekiguchi, COCOON2008);

$$\bar{x}(r) = \begin{cases} 
\frac{m}{(\alpha-m)} \sqrt{\frac{3r-m}{2(r-m)}} \sqrt{\frac{f(r)}{r}} + \frac{f(r)}{2} \sqrt{\frac{r}{f(r)}} & , \ r \in [\alpha, \beta]; \\
0, & \ r \notin [a, \beta],
\end{cases}$$

where the constants $\alpha, \beta$ satisfy $\beta(\beta-m)f(\beta) = \int_{\beta}^{M} r f(r)dr, \int_{\alpha}^{\beta} \bar{x}(r)dr = 1$.

Second, we discuss theoretical properties of metric regularity. The following is the classical theorem on matrices:

$$\inf_{B \in \mathbb{R}^{n \times n}} \{ \| B \| \mid A + B \text{ is singular} \} = \frac{1}{\| A \|}$$

This formula enables us to measure a robustness of regularity of matrix equalities. It was extended to general set-valued mappings between Euclidean spaces using metric regularity. However it is known that the formula does not holds in Banach spaces generally. We prove a version
of the theorem in continuous function spaces. Let $\Omega$ be a compact Hausdorff space and $X = C(\Omega)^n, Y = C(\Omega)^m$. Define a set-valued mapping $F: X \rightrightarrows Y$ by

$$F(x) \ni y \iff f(x(\omega)) \leq y(\omega),$$

where $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable, and the order is coordinatewise. Metric regularity of $F$ represents stability of the solution mapping for the inequality system.

**Theorem** (Ioffe-Sekiguchi, Math. Program. 2009). For a point $(\bar{x}, \bar{y})$ in the graph of $F$,

$$\inf_{g: X \to Y} \{ \text{Lip} \ g(\bar{x}) \mid F + g \ \text{is not metrically regular at} \ (\bar{x}, \bar{y} + g(\bar{x})) \} = \frac{1}{\text{reg } F(\bar{x}|\bar{y})},$$

where $\text{Lip} \ g$ is the Lipschitz constant of $g$ at $\bar{x}$.

Faculty of Marine Technology, Tokyo University of Marine Sciences and Technology, 2-1-6 Etchujima, Koto, Tokyo, Japan

E-mail address: yoshi-s@kaiyodai.ac.jp