A remark on Rademacher's theorem

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Rademacher (Ann. Math. 79) proved the following.

Theorem 1 *Let D be an open set in $\mathbf{R}^N$, and $F : D \to \mathbf{R}^N$ be a continuous function.*
(1) *If*

$$\limsup_{h\to 0} \frac{1}{|h|}|F(x+h) - F(x)| < \infty \qquad \text{for a.e.} x \in D,$$

then F is totally differentiable at a.e.$x \in D$.
(2) *Moreover, if $F : D \to \mathbf{R}^N$ is injective, then*

$$\int_A |det\nabla F(x)|dx = |F^{-1}(A)|$$

for any Borel subset $A \subset D$.

As a corollary to this theorem, we have the following.

Corollary 2 *Let D be an open set in $\mathbf{R}^N$, and $F : D \to \mathbf{R}^N$ be a injective Lipschitz continuous function. Then F is totally diferebtiable at a.e.$x \in D$, and*

$$\int_A |det\nabla F(x)|dx = |F^{-1}(A)|$$

for any Borel subset $A \subset D$.

This classical theorem is fine, but sometimes it is difficult to check whether a map is injective.

On the other hand, we have the following by using Sard's theorem.

Theorem 3 *Let $F : \mathbf{R}^N \to \mathbf{R}^N$ be a continuously differentiable function. Then for any nonnegative measureble functions f and g defined in $\mathbf{R}^N$*

$$\int_{\mathbf{R}^N} f(x)g(F(x))|det\nabla F(x)|dx = \int_{\mathbf{R}^N} g(y)N(y;f)dy,$$

where

$$N(y;f) = \sum_{x\in F^{-1}(y)} f(x), \qquad y \in \mathbf{R}^N.$$

We prove the following.

Theorem 4 *Let $F : \mathbf{R}^N \to \mathbf{R}^N$ be a continuous function belonging to $W^1_{p,loc}(\mathbf{R}^N;\mathbf{R}^N)$ for some $p > N$. Then there exists a $N : \mathcal{B}(\mathbf{R}^N) \times \mathbf{R}^N \to \{0, 1, 2, \ldots, \infty\}$ satisfying the following.*
(1) *$N(\cdot, x) : \mathcal{B}(\mathbf{R}^N) \to \{0, 1, 2, \ldots, \infty\}$ is a measure for any $x \in \mathbf{R}^N$.*
(2) *$N(A, \cdot) : \mathbf{R}^N \to \{0, 1, 2, \ldots, \infty\}$ is measureble for any $A \in \mathcal{B}(\mathbf{R}^N)$.*
(3) *$N(\mathbf{R}^N \setminus F^{-1}(x), x) = 0$ for any $x \in \mathbf{R}^N$.*
(4) *For any nonnegative measureble functions f and g defined in $\mathbf{R}^N$*

$$\int_{\mathbf{R}^N} f(x)g(F(x))|det\nabla F(x)|dx = \int_{\mathbf{R}^N} g(y)(\int_{\mathbf{R}^N} f(z)N(dz;y))dy.$$