A remark on Rademacher’s theorem
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Rademacher (Ann. Math. 79) proved the following.

**Theorem 1** Let $D$ be an open set in $\mathbb{R}^N$, and $F : D \to \mathbb{R}^N$ be a continuous function.

1. If
   \[ \limsup_{h \to 0} \frac{1}{|h|} |F(x + h) - F(x)| < \infty \quad \text{for a.e.} \ x \in D, \]
   then $F$ is totally differentiable at a.e. $x \in D$.

2. Moreover, if $F : D \to \mathbb{R}^N$ is injective, then
   \[ \int_A |\det \nabla F(x)| dx = |F^{-1}(A)| \]
   for any Borel subset $A \subset D$.

As a corollary to this theorem, we have the following.

**Corollary 2** Let $D$ be an open set in $\mathbb{R}^N$, and $F : D \to \mathbb{R}^N$ be a injective Lipschitz continuous function. Then $F$ is totally differentiable at a.e. $x \in D$, and
   \[ \int_A |\det \nabla F(x)| dx = |F^{-1}(A)| \]
   for any Borel subset $A \subset D$.

This classical theorem is fine, but sometimes it is difficult to check whether a map is injective.

On the other hand, we have the following by using Sard’s theorem.

**Theorem 3** Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuously differentiable function. Then for any nonnegative measurable functions $f$ and $g$ defined in $\mathbb{R}^N$
   \[ \int_{\mathbb{R}^N} f(x) g(F(x)) |\det \nabla F(x)| dx = \int_{\mathbb{R}^N} g(y) N(y; f) dy, \]
   where
   \[ N(y; f) = \sum_{x \in F^{-1}(y)} f(x), \quad y \in \mathbb{R}^N. \]

We prove the following.

**Theorem 4** Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function belonging to $W^{1,p,\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ for some $p > N$. Then there exists a $N : \mathcal{B}(\mathbb{R}^N) \times \mathbb{R}^N \to \{0,1,2,\ldots,\infty\}$ satisfying the following.

1. $N(\cdot, x) : \mathcal{B}(\mathbb{R}^N) \to \{0,1,2,\ldots,\infty\}$ is a measure for any $x \in \mathbb{R}^N$.
2. $N(A, \cdot) : \mathbb{R}^N \to \{0,1,2,\ldots,\infty\}$ is measurable for any $A \in \mathcal{B}(\mathbb{R}^N)$.
3. $N(\mathbb{R}^N \setminus F^{-1}(x), x) = 0$ for any $x \in \mathbb{R}^N$.
4. For any nonnegative measurable functions $f$ and $g$ defined in $\mathbb{R}^N$
   \[ \int_{\mathbb{R}^N} f(x) g(F(x)) |\det \nabla F(x)| dx = \int_{\mathbb{R}^N} g(y) (\int_{\mathbb{R}^N} f(z) N(dz; y)) dy. \]