EXISTENCE OF MOUNTAIN PASS TYPE HOMOCLINIC SOLUTIONS FOR A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. Let $N \geq 3$ and $\mathcal{D} \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. In this paper, we consider the existence and multiplicity of homoclinic solutions for nonlinear elliptic problem

$$-\Delta u + g(x, u) = 0,$$

where $g \in C^1(\mathbb{R} \times \mathcal{D}, \mathbb{R}^N)$ has a spacially periodicity.

1. Introduction

Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a cylindrical domain, i.e., $\Omega = \mathbb{R} \times \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^{N-1}$ is a bounded open domain with a smooth boundary. In the present paper, we consider the existence of homoclinic solutions of boundary value problem

$$\text{(P)} \qquad \begin{cases} -\Delta u + g(x, u) & = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $g \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $\nu = \nu(y)$ denotes the outward pointing normal derivative to $\partial\mathcal{D}$. For $x \in \Omega$, we set $x = (x_1, y)$, where $x_1 \in \mathbb{R}$ and $y \in \mathcal{D}$. We impose the following conditions on g :

(g1) $\qquad g(x, z) \in C^1(\mathbb{R} \times \overline{\Omega}, \mathbb{R})$ and is $1-$periodic with respect to x_1;

(g2) $\qquad G(x, z) = \int_0^z g(x, \tau) d\tau$ is $1-$periodic with respect to z.

In [2] and [3], Rabinowitz considered the existence of spacially heteroclinic solutions of problem (P) under the assumption (g1), (g2) and additional coniditon

(g3) $\qquad g(x, z)$ is even with respect to $x_1 \in \mathbb{R}$.

In [5], the exsitence of the heteroclinic solutions of (P) was established without the evenness condition (g3). Recently, using the results in these papers, the existence of homoclinic solutions of (P) was established in [4].

The purpose of this paper is to investigate the existence and multiplicity of homoclinic solutions of (P). We will show that there is a sequence of homoclinic solutions of (P) such that each solution is given as a local minimal of corresponding funcitional to (P). We also show the exsitence of a sequence

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of homoclinic solutions of (P) such that each solution is a moutain pass type solution for (P).

2. Preliminaries and Statements of Main Results

Throughout the rest of this paper, we assume that coniditions (g1) and (g2) hold. For $x, y \in \mathbb{R}^N$, we denote by $x \cdot y$ the inner product of x and y. For each $n \geq 1$ and each open set $U \subset \mathbb{R}^n$, we denote by $\|\cdot\|_{H^1(U)}$ and $\|\|_{L^2(U)}$ the norm of $H^1(\Omega)$ and $L^2(\Omega)$ defined by $\|u\|^2_{H^1(U)} = \int_U |\nabla u|^2 \, dx$ and $\|v\|^2 = \int_U |v|^2 \, dx$ for each $u \in H^1(U)$ and $v \in L^2(U)$, respectively. We denote by $\langle \cdot, \cdot \rangle_U$ the inner product of $H^1(U)$. Put $\Omega_i = [i, i+1] \times \mathcal{D}$ for each $i \in \mathbb{Z}$.For each function $u : H^1_{loc}(\Omega) \longrightarrow \mathbb{R}$ and $m \in \mathbb{Z}$, we denote by $u(m)$ the restrection of u on $H^1(\Omega_m)$. Let $v \in H^1_{loc}(\Omega)$ and $t \in \mathbb{R}$. We denote by $\tau_t v$ the function defined by

$$\tau_t v(x_1, y) = v(x_1 - t, y) \qquad \text{for all } (x_1, y) \in \mathbb{R} \times \mathcal{D}.$$

We set

$$L(u)(x) = \frac{1}{2} |\nabla u(x)|^2 - G(x, u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } x \in \Omega.$$

Put

$$I_i(u) = \int_{\Omega_i} L(u) dx \qquad \text{for } i \in \mathbb{Z} \text{ and } u \in H^1(\Omega_i)$$

and

$$E = \left\{ u \in W^{1,2}(\Omega_0) : u \text{ is one periodic in } x_1 \right\}.$$

We put

$$c_0 = \inf_{u \in E} I(u) \quad \text{ and } \quad M_0 = \{ u \in E : I_0(u) = c_0 \}.$$

Then the following is known.

Proposition 1 ([3]). *$M_0 \neq \emptyset$ and M_0 is an ordered set.*

Here we put

$$a_j(u) = \int_{\Omega_j} L(u) dx - c_0 \qquad \text{for } j \in \mathbb{Z} \text{ and } u \in H^1(\Omega_j),$$

and

$$J_{l,m}(u) = \sum_{j=l}^{m} a_j(u) \quad \text{for } l, m \in \mathbb{Z} \text{ with } l \leq m.$$

We also put

$$J(u) = \liminf_{l \longrightarrow -\infty} J_{l,0}(u) + \liminf_{m \longrightarrow \infty} J_{0,1}(u) \qquad \text{for } u \in H^1_{loc}(\Omega),$$

$$J_{-\infty,m}(u) = \liminf_{l \longrightarrow -\infty} J_{l,0}(u) + J_{1,m}(u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } m \geq 1,$$

$$J_{m,\infty}(u) = J_{m,0}(u) + \liminf_{l \longrightarrow -\infty} J_{1,l}(u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } m \leq 0.$$

For each $v, w \in M_0$ with $v < w$, we set

$$[v, w] = \left\{ u \in H^1(\Omega) : v \leq u \leq w \right\}, \quad [v, w]_m = \left\{ u|_{\Omega_m} : u \in [v, w] \right\}$$

$$\Gamma_-(z) = \left\{ u \in [v, w] : J(u) < \infty, \|u - z\|_{L^2(\Omega_j)} \longrightarrow 0, \text{ as } j \longrightarrow -\infty \right\} \text{ for } z \in \{v, w\},$$

$$\Gamma_+(z) = \Big\{u \in [v,w] : J(u) < \infty, \|u - z\|_{L^2(\Omega_j)} \longrightarrow 0, \text{ as } j \longrightarrow \infty\Big\} \text{ for } z \in \{v,w\}.$$

$$\Gamma(z_1,z_2) = \Gamma_-(z_1) \cap \Gamma_+(z_2) \qquad \text{for } z_1, z_2 \in \{v,w\}\,.$$

and

$$\Gamma(z,z) = \Big\{u \in [v,w] : \overline{J}(u) < \infty, \|u - v\|_{L^2(\Omega_j)} \longrightarrow 0, \text{ as } j \longrightarrow \pm\infty\Big\} \text{ for } z \in \{v,w\}\,.$$

Then we have

Proposition 2 (cf. [4, 5])**.** *For each* $v, w \in M_0$ *and* $u \in \Gamma(v,w)$,, $\lim_{l \longrightarrow -\infty} J_{l,0}(u)$ *and* $\lim_{m \longrightarrow \infty} J_{1,m}(u)$ *exists.*

Remark 1. *From Proposition 2, it follows that for each* $u \in \Gamma_-(v)$

$$J_{-\infty,m}(u) = \lim_{l \longrightarrow -\infty} J_{l,0}(u) + J_{1,m}(u) \qquad \textit{for } m \geq 1.$$

Similarly, we have for each $u \in \Gamma(w)$,

$$J_{m,\infty}(u) = J_{m,0}(u) + \lim_{l \longrightarrow \infty} J_{1,l}(u) \qquad \textit{for } m \leq 0.$$

Let $v, w \in M_0$. We call $u \in H^1_{loc}(\Omega)$ a heteroclinic solution of (P) in $[v,w]$ if $u \in \Gamma(v,w)$ and u is a solution of (P). A solution $u \in H^1_{loc}(\Omega)$ of (P) is called a homoclinic solution in $[v,w]$ if $u \in \Gamma(v,v)$ or $u \in \Gamma(w,w)$.

We put

$$c(v,w) = \inf_{u \in \Gamma(v,w)} J(u), \quad \text{for } v, w \in M_0$$

and

$$\mathcal{M}(v,w) = \{u \in \Gamma(v,w) : J(u) = c(v,w)\} \qquad \text{for } v, w \in M_0.$$

Then we have

Proposition 3 ([2])**.** *For each* $v, w \in M_0$ *which are adjacent and* $v < w$, $\mathcal{M}(v,w)$ *is a nonempty ordered set.*

We will consider the existence of homoclinic solution of (P) under the following conditions:

(*) M_0 has adjacent elements v, w with $v < w$.

(**) $\mathcal{M}(v,w), \mathcal{M}(w,v)$ are not continimum.

(C) $\inf\big\{I(v) : v \in H^1(\Omega_0)\big\} \geq c_0$.

We can now state our main results:

Theorem 1. *Assume that* (g*1*), (g*2*), $(*), (**)$ *and* (*C*) *holds. Let* $v, w \in M_0$ *be adjacent with* $v < w$*. Let* $v_1, v_2 \in \mathcal{M}(v,w)$ *be adjacent with* $v_1 < v_2$*. Then there exists a positive integer* n_0 *and a sequence* $\{u_n\} \subset \Gamma(v,v)$ *of homoclinic solutions of* (*P*) *such that*

(1) $u_n < u_{n+1}$ *for each* $n \geq 1$:

(2) $\tau_{-n_0-n} v_1[0] \leq u_n[0] \leq \tau_{-n_0-n-1} v_2[0]$ *for each* $n \geq 1$:

(3) $\lim_{n \longrightarrow \infty} J(u_n) = c(v,w) + c(w,v)$.

Remark 2. *In* [4]*, the existence of homoclinic solutions of* (P) *was established without assuming the condition* (C)*. Assuming* (C)*, we can get shaper characterization of solutions for* (P)*. The condition* (C) *is satisfied if the functions satsifies* $(g3)$ $(cf.$ [2]$)$

Theorem 2. *Assume that* $(g1), (g2), (*), (**)$ *and* (C) *holds. Let* $v, w \in M_0$ *such that* v, w *are adjacent and* $v < w$*. Let* $v_1, v_2 \in \mathcal{M}(v, w)$ *be adjacent with* $v_1 < v_2$*. There exists* $\delta_0 > 0$ *, a posive integer* n_0 *and a sequence* $\{u_n^*\} \subset \Gamma(v, v)$ *of homoclinic solutions of* (P) *such that*

(1) $u_n^* < u_{n+1}^*$ *for each* $n \geq 1$:

(2) $\tau_{-n_0-n}v_1[0] < u_n^*[0] \leq \tau_{-n_0-n-1}v_2[0]$ *for each* $n \geq 1$:

(3) $\liminf_{n \longrightarrow \infty} J(u_n^*) > c(v, w) + c(w, v) + \delta_0$.

Remark 3. *The solutions obtained in Theorem 2 are of moutain pass type. The existence of mountain pass type solutions for heteroclinic solutions of (P) was considered in* [1]*.*

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