Saddle Points and Nonlinear Operators

Wataru Takahashi

TOKYO INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES

Let H be a Hilbert space and let g be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then, we consider the convex minimization problem:

$$\min\{g(x) : x \in H\}. \tag{$*$}$$

For such a g, we can define a multivalued operator ∂g on H by

$$\partial g(x) = \{x^* \in H : g(y) \geq g(x) + (x^*, y - x), y \in H\}$$

for all $x \in H$. Such a ∂g is said to be the *subdifferential* of g. An operator $A \subset H \times H$ is *accretive*, if for $(x_1, y_1), (x_2, y_2) \in A$,

$$(x_1 - x_2, y_1 - y_2) \geq 0.$$

If A is accretive, we can define, for each positive λ, the resolvent $J_\lambda : R(I + \lambda A) \to D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$. We know that J_λ is a nonexpansive mapping. An accretive operator $A \subset H \times H$ is called *m-accretive*, if $R(I + \lambda A) = H$ for all $\lambda > 0$. An m-accretive operator is equivalent to a maximal monotone operator in a Hilbert space. If $g : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then ∂g is an m-accretive operator.

We know that one method for solving $(*)$ is the *proximal point algorithm* first introduced by Martinet. The proximal point algorithm is based on the notion of resolvent J_λ, i.e.,

$$J_\lambda x = \arg\min\left\{g(z) + \frac{1}{2\lambda}\|z - x\|^2 : z \in H\right\},$$

introduced by Moreau. The proximal point algorithm is an iterative procedure, which starts at a point $x_1 \in H$, and generates recursively a sequence $\{x_n\}$ of points $x_{n+1} = J_{\lambda_n} x_n$, where $\{\lambda_n\}$ is a sequence of positive numbers, see, for instance, Rockafellar.

On the other hand, Halpern and Mann introduced the following iterative schemes to approximate a fixed point of a nonexpansive mapping T of H into itself:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \quad \text{for } n \geq 1$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n \quad \text{for } n \geq 1,$$

respectively, where $x_1 = x \in H$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In this talk, motivated by their iterative methods for approximation of fixed points, we first prove weak and strong convergence theorems for resolvents of maximal monotone operators in a Hilbert space and a Banach space. Next, using these results, we consider the problem of finding a saddle point of a two variable function in a Hilbert space and a Banach space.