Differentiability and bifurcation

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Abstract

This lecture concerns the relation between elementary notions of differentiability and bifurcation. It reports on joint work done in collaboration with Gilles Evequoz at ELFL, Lausanne.

Consider a function $F : X \to Y$ where X and Y are real Banach spaces. The two most common notions of differentiability of F are those named after Gâteaux and Fréchet, but many variants exist. One of these is differentiability in the sense of Hadamard which we express in the following form:
$F : X \to Y$ is *Hadamard differentiable* at $u \in X$
if there exists a bounded linear operator $T = F'(0) : X \to Y$ such that

$$\begin{aligned} \lim_{t_n \to 0} \frac{F(u + t_n v_n) - F(u)}{t_n} &= Tv \text{ for all } v \in X \\ \text{for all } \{t_n\} &\subset \mathbb{R}\backslash\{0\} \text{ with } t_n \to 0 \\ \text{and for all } \{v_n\} &\subset X \text{ with } v_n \to v. \end{aligned}$$

It is easily seen that Fréchet differentiability implies Hadamard differentiability which in turn implies Gâteaux differentiability. But Gâteaux differentiable functions may fail to be Hadamard differentiable and Hadamard differentiable functions may fail to be Fréchet differentiable. However, if $\dim X < \infty$, then Hadamard and Fréchet differentiability coincide.

In its simplest form, abstract bifurcation theory deals with equations of the form

$$F(u) = \lambda u \text{ where } \lambda \in \mathbb{R},\ X \subset Y \text{ and } F(0) = 0. \tag{1}$$

A point $\lambda \in \mathbb{R}$ is called a *bifurcation point* for (1) (from the line of trivial solutions $\mathbb{R} \times \{0\} \subset \mathbb{R} \times X$) if there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times X$

having the following properties

$$\begin{aligned} F(u_n) &= \lambda_n u_n \text{ and } u_n \neq 0 \text{ for all } n \in \mathbb{N}, \\ \lambda_n &\to \lambda \text{ and } \|u_n\|_X \to 0 \text{ as } n \to \infty. \end{aligned}$$

Let $B_F \subset \mathbb{R}$ denote the set of all bifurcation points for (1). The most basic result in classical bifurcation theory states that, if F is Fréchet differentiable at $u = 0$, then $B_F \subset \sigma(F'(0))$ where

$$\sigma(T) = \{\lambda \in \mathbb{R} : T - \lambda I : X \to Y \text{ is not an isomorphism}\}$$

denotes the spectrum of a linear operator $T : X \to Y$. One then seeks additional assumptions on F and $\lambda \in \sigma(F'(0))$ that ensure that $\lambda \in B_F$. These results have important applications in many fields.

When concrete problems involving differential or functional equations are expressed in the form (1), it is not always the case that F is Fréchet differentiable at 0. Sometimes Hadamard differentiability is the best that can be obtained and, in such cases, we may have $B_F \not\subset \sigma(F'(0))$

In this lecture, a series of basic examples and results concerning these situations will be presented.