Cooperative Extensions of the Bayesian Game*

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1 Basic Ingredients

1.1 One-shot model

The required model needs to embody

$$\begin{cases} \text{the Bayesian game} \\ \text{the non-side-payment game (NTU game).} \end{cases}$$

NOTATION:

N: finite set of players.
$\mathcal{N} := 2^N \setminus \{\emptyset\}$: nonempty coalitions.
C^j: choice set (action set).
T^j: type set.
$C^S := \prod_{j\in S} C^j$, $T^S := \prod_{j\in S} T^j$,
$C := C^N$, $T := T^N$.
$u^j : C \times T \to \mathbf{R}$: v N-M utility function.
ex ante, *interim* (*in mediis*), *ex post*.
$\pi^j(\cdot \mid t^j)$: prob. on $T^{N\setminus\{j\}}$, given t^j.
$\pi^j(t^{N\setminus\{j\}} \mid t^j) = \frac{\pi^j(t^{N\setminus\{j\}}, t^j)}{\pi^j(T^{N\setminus\{j\}} \times \{t^j\})}$, if $\exists\ \pi^j$ on T.

DEF. (Harsanyi, 1967/1968) A *Bayesian game* is a list of specified data,

$$\{C^j,\ T^j,\ u^j,\ \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}\}_{j \in N}.$$

NOTATION

$\mathbf{C}_0^S(t)(\subset C^S)$: feasible joint choices

Notice: $\begin{cases} \mathbf{C}_0^S(t) \neq \Pi_{j\in S}\, \mathbf{C}_0^j(t) \\ \mathbf{C}_0^S(t) \neq \mathbf{C}_0^S(t') \text{ if } t \neq t' \end{cases}.$

DEF. *Complete information* $\Leftrightarrow \#T = 1$.

If, further, $u^j(c) = u^j(c^j)$

$$V(S) := \left\{ u \in \mathbf{R}^N \;\middle|\; \begin{array}{l} \exists\ c^S \in \mathbf{C}_0^S : \\ \forall\ j \in S : u_j \leq u^j(c^j) \end{array} \right\}.$$

DEF. (Aumann and Peleg, 1960) A *non-side-payment game* is a cylinder-valued correspondence $V : \mathcal{N} \to \mathbf{R}^N$.

NOTATION

$\mathcal{T}^S$: the info. structure on T generated by
$\{\{t^S\} \times T^{N \setminus S} \mid t^S \in T^S\}$.

$\mathcal{T}^j := \mathcal{T}^{\{j\}}$: private info. structure.

$T(\pi^j) := \underset{t^j \in T^j}{\cup} [\{t^j\} \times \text{ supp } \pi^j(\cdot \mid t^j)]$.

$\mathcal{T}^j(\pi^j) := \mathcal{T}^j \vee \{\emptyset,\ T(\pi^j),\ T \setminus T(\pi^j),\ T\}$.

STRATEGY, STRATEGY BUNDLE

$T(S) :=$ the domain of strategy bundles for S, arb. given, s.t. $\underset{t^j \in T^j}{\cup} T(\pi^j) \subset T(S) \subset T$.

$X^j(S) := \{x^j : T(S) \to C^j\}$, j's *strategies*.

$X^S := \underset{j \in S}{\Pi} X^j(S)$, $X := X^N$.

$F^S : X \to X^S$, feasible strategy corresp.

$F^S(\bar{x}) \subset \{\mathcal{T}^S\text{-meas'ble selections of } \mathbf{C}_0^S|_{T(S)}\}$

DEF. (Ichiishi and Idzik, 1996) A *Bayesian society* is a list of specified data

$$\mathcal{S} := \begin{pmatrix} \{C^j, T^j, u^j\}_{j \in N}, \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}, \\ \{\mathbf{C}_0^S, T(S), F^S\}_{S \in \mathcal{N}} \end{pmatrix}.$$

1.2 Example

EX. A *Bayesian pure exchange economy* is a list of specified data,

$$\mathcal{E}_{pe} := \{T^j, \mathbf{R}^l_+, u^j, e^j, \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}\}_{j \in N}.$$

Def. of the associated Bayesian society,

$$\mathcal{S} := \begin{pmatrix} \{C^j, T^j, u^j\}_{j \in N}, \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}, \\ \{\mathbf{C}^S_0, T(S), F^S\}_{S \in \mathcal{N}} \end{pmatrix}.$$

N, T^j, u^j and $\pi^j(\cdot \mid t^j)$: given in economy $\mathcal{E}_{pe}$.
$C^j := \mathbf{R}^l$,
$T(S)$: arbitrarily given,
$F^S(\bar{x}) :=$ attainable excess demand plans $=$

$$\left\{ z^S : T(S) \to \mathbf{R}^{l \cdot |S|} \;\middle|\; \begin{array}{l} z^S \text{ is } \mathcal{T}^S\text{-measurable}, \\ \forall\, t : \forall\, j : z^j(t) + e^j(t^j) \geq \mathbf{0}, \\ \forall\, t : \sum_{j \in S} z^j(t) \leq \mathbf{0} \end{array} \right\}.$$

Some works formulate the model so that j's strategy is a demand plan,

$$x^j : t \mapsto z^j(t) + e^j(t^j).$$

Choice of excess demand plan versus demand plan as a strategy affects some results.

1.3 Two formulations of incomplete information

1. Harsanyi's *type-space approach*:

$$\left\{T^j, \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}\right\}_{j \in N}$$

Notice $\mathcal{T}^i \wedge \mathcal{T}^j = \{\emptyset, T\}$ if $i \neq j$.

2. General *state-space approach*:

$$\left(\Omega, \mathcal{F}^j, \{\pi^j(\cdot \mid F)\}_{F \in \mathcal{P}^j}\right),$$

where $\mathcal{P}^j :=$ the min'l elts of $\mathcal{F}^j =$ partition.
Possibility: $\mathcal{F}^i \wedge \mathcal{F}^j \supsetneqq \{\emptyset, \Omega\}$ even if $i \neq j$.

Jackson's (1991) observations:

1 $\Rightarrow$ 2.

Given $\left\{T^j, \{\pi^j(\cdot \mid t^j)\}_{t^j \in T^j}\right\}_{j \in N}$, define:
$\Omega := T$, $\mathcal{F}^j := \mathcal{T}^j$.

2 $\Rightarrow$ 1, under the ass. wlog, $\vee_{i \in N} \mathcal{F}^i = 2^\Omega$.

Given $\left(\Omega, \mathcal{F}^j, \{\pi^j(\cdot \mid F)\}_{F \in \mathcal{P}^j}\right)$, define:
$T^j := \mathcal{P}^j$.

Observe:

$$\forall\ \omega : \exists!\ \{F^j\}_{j\in N} \in \Pi_{j\in N}\,\mathcal{P}^j : \{\omega\} = \cap_{j\in N}\, F^j.$$
$$\forall\ \{F^j\}_{j\in N} : \#(\cap_{j\in N}\, F^j) = 0 \text{ or } 1.$$

If $\cap_{j\in N}\, F^j \neq \emptyset$, identify

$$\{F^j\}_{j\in N} \in \prod_{j\in N} T^j \text{ and } \bigcap_{j\in N} F^j \in \Omega.$$

EX. $N = \{1,2\}$, $\Omega = \{a,b,c\}$,
$\mathcal{P}^1 = \{\{a\},\{b,c\}\}$, $\mathcal{P}^2 = \{\{a,b\},\{c\}\}$.

$\{a,b\}$	$\{a\}$	$\{b\}$
$\{c\}$	$\emptyset$	$\{c\}$
	$\{a\}$	$\{b,c\}$

.

EX. $N = \{1,2\}$, $\Omega = \{a,b,c\}$,
$\mathcal{P}^j = \{\{a\},\{b\},\{c\}\}$, $j = 1,2$.
(i.e., the *interim* period = the *ex post* period)

$\{c\}$	$\emptyset$	$\emptyset$	$\{c\}$
$\{b\}$	$\emptyset$	$\{b\}$	$\emptyset$
$\{a\}$	$\{a\}$	$\emptyset$	$\emptyset$
	$\{a\}$	$\{b\}$	$\{c\}$

.

1.4 Measurability as a feasibility requirement

Suppose that
N is entertaining $\bar{x} : T(N) \to C$, but that
S may defect and take $x^S : T(S) \to C^S$.

DEF. The *private information case*: At the time of action (strategy execution), j has only $\mathcal{T}^j$, so knows only his true type t^j and the *interim* probability $\pi^j(\cdot \mid t^j)$.

CONDITION (Radner, 1967; Yannelis, 1991) In the private information case, members of S agree only on *private measurable* strategies $x^S \in F^S(\bar{x})$ in that x^j is $\mathcal{T}^j$-measurable for every $j \in S$.

$$F'^S(\bar{x}) := \left\{ x^S \in F^S(\bar{x}) \;\middle|\; \begin{array}{l} \forall\ j \in S : \\ x^j \text{ is } \mathcal{T}^j\text{-measurable.} \end{array} \right\}.$$

1.5 Bayesian incentive compatibility: private info case

(Abuse of notation:
$\pi^j(\cdot \mid \bar{t}^j)$, defined on T rather than on $T^{N\setminus\{j\}}$)

Suppose that in the private information case N is entertaining $\bar{x} : T(N) \to C$, but that S may defect and take $x^S : T(S) \to C^S$.

$\{\bar{t}^j\}_{j\in S}$: S's true type profile.
j's honest action $x^j(\bar{t}^j) \Rightarrow Eu^j(x^S, \bar{x}^{N\setminus S} \mid \bar{t}^j)$.
j's wrong action $c^j \in x^j(T(S)) \setminus \{x^j(\bar{t}^j)\}$

$$\Rightarrow Eu^j(c^j, x^{S\setminus\{j\}}, \bar{x}^{N\setminus S} \mid \bar{t}^j).$$

Two types of wrong action:

1. conservative attitude:

$$c^j \in x^j\left(\bigcap_{i\in S\setminus\{j\}} \text{supp}\ \pi^i(\cdot \mid \bar{t}^i)\right).$$

2. bold attitude:

$$c^j \in x^j\left(\bigcup_{i\in S\setminus\{j\}} \text{supp}\ \pi^i(\cdot \mid \bar{t}^i)\right).$$

CONDITION (d'Aspremont and Gérard-Varet, 1979) In the private information case, members of S agree only on those strategies $x^S \in F^S(\bar{x})$ that are *Bayesian incentive-compatible*, that is,

$$\forall\ j \in S : \forall\ \bar{t} \in T(S) : \forall\ c^j \in x^j \left(\bigcap_{i \in S \setminus \{j\}} \text{supp } \pi^i(\cdot \mid \bar{t}^i) \right) :$$
$$Eu^j(x^S, \bar{x}^{N \setminus S} \mid \bar{t}^j) \geq Eu^j(c^j, x^{S \setminus \{j\}}, \bar{x}^{N \setminus S} \mid \bar{t}^j).$$

CONDITION (d'Aspremont and Gérard-Varet, 1979) In the private information case, members of S agree only on those strategies $x^S \in F^S(\bar{x})$ that are *strongly Bayesian incentive-compatible*, that is,

$$\forall\ j \in S : \forall\ \bar{t} \in T(S) : \forall\ c^j \in x^j \left(\bigcup_{i \in S \setminus \{j\}} \text{supp } \pi^i(\cdot \mid \bar{t}^i) \right) :$$
$$Eu^j(x^S, \bar{x}^{N \setminus S} \mid \bar{t}^j) \geq Eu^j(c^j, x^{S \setminus \{j\}}, \bar{x}^{N \setminus S} \mid \bar{t}^j).$$

$$\hat{F}^S(\bar{x}) := \left\{ x^S \in F'^S(\bar{x}) \;\middle|\; \begin{array}{l} x^S: \text{ Bayesian} \\ \text{incentive-compatible.} \end{array} \right\}.$$

PROP. (Hahn and Yannelis, 1997)
$\mathcal{E}_{pe}$: the Bayesian pure exchange economy in the private information case.
j's strategy: j's excess demand plan z^j.
the coalitional feasibility:

$$\forall\ t \in T(S) : \sum_{j \in S} z^j(t) = \mathbf{0}.$$

Then,
private measurability
$\Rightarrow$ Bayesian incentive compatibility.

REMARK See Ichiishi and Radner (1999) for "$\leq$" $\Rightarrow$ "$=$".

REMARK This proposition is no longer valid if a demand plan x^j is used as a strategy.
Ex. $l = 1$, $T^j = \{a^j, b^j\}$, $u^j(c^j, t) = c^j$,

$$e^j(t^j) = \begin{cases} 1, & \text{if } t^j = a^j, \\ 2, & \text{if } t^j = b^j. \end{cases}$$

REMARK The proposition is not valid in the general model of Bayesian society $\mathcal{S}$.

REMARK Private measurability does not imply strong Bayesian incentive compatibility.
Ex. $l = 1$, $N = \{1, 2, 3\}$.

$$
\begin{aligned}
T^1 &= \{t^1_a, t^1_{bc}\},\\
T^2 &= \{t^2_{ab}, t^2_c\},\\
T^3 &= \{t^3_a, t^3_b, t^3_c\},\\
\text{supp } \pi^1(\cdot \mid t^1_h) &= \begin{cases} \{(t^1_a, t^2_{ab}, t^3_a)\}, \text{ if } h = a,\\ \{(t^1_{bc}, t^2_{ab}, t^3_b), (t^1_{bc}, t^2_c, t^3_c)\}, \text{ if } h = bc, \end{cases}\\
\text{supp } \pi^2(\cdot \mid t^2_h) &= \begin{cases} \{(t^1_a, t^2_{ab}, t^3_a), (t^1_{bc}, t^2_{ab}, t^3_b)\}, \text{ if } h = ab,\\ \{(t^1_{bc}, t^2_c, t^3_c)\}, \text{ if } h = c, \end{cases}\\
\text{supp } \pi^3(\cdot \mid t^3_h) &= \begin{cases} \{(t^1_a, t^2_{ab}, t^3_a)\}, \text{ if } h = a,\\ \{(t^1_{bc}, t^2_{ab}, t^3_b)\}, \text{ if } h = b,\\ \{(t^1_{bc}, t^2_c, t^3_c)\}, \text{ if } h = c, \end{cases}\\
T(S) &= \{(t^1_a, t^2_{ab}, t^3_a), (t^1_{bc}, t^2_{ab}, t^3_b), (t^1_{bc}, t^2_c, t^3_c)\},\\
u^j(c^j, t) &= c^j,\\
e^j(t^j) &= 2, \text{ for all } t^j \in T^j.
\end{aligned}
$$

1.5 Bayesian inc. compatibility (cont'd): mediator-based case

The role of a *mediator* (enforcement agency):

1. S designs excess demand plan $z^S \in F^S$.
2. Player j confidentially reports t^j to the mediator.
3. The mediator has reports t^S.
4. The mediator tells j to make choice $z^j(t^S)$.

Let $\bar{t}^S$ be the true type profile.

Honest report $\Rightarrow Eu^j(z^j + e^j \mid \bar{t}^j)$.

Dishonest report $\Rightarrow Eu^j(z^j(\tilde{t}^j, \cdot) + e^j \mid \bar{t}^j)$.

CONDITION (Vohra, 1999) Strategy bundle $z^S \in F^S$ is Bayesian incentive-compatible, in the sense that

$$\neg \, \exists \, j \in S : \exists \, \bar{t}^j : \exists \, \tilde{t}^j : \quad Eu^j(z^j(\tilde{t}^j, \cdot) + e^j \mid \bar{t}^j) > Eu^j(z^j + e^j \mid \bar{t}^j).$$

Problem: No mediator in reality.

Alternative scenario (to eliminate the mediator):

1. S designs excess demand plan $z^S \in F^S$.
2. Players independently and simultaneously report t^j's each other.
3. The players have updated information t^S.
4. Player j makes the promised choice $z^j(t^S)$.

Problem:

$$\begin{cases} \text{step 2 – decision at the } \textit{interim} \text{ stage} \\ \text{step 4 – decision at the } \textit{ex post} \text{ stage} \end{cases}$$

EX. $l = 1$, $\#N = \#T^j = 2$, $e^j(t^j) = 1$, $u^j(c^j, t) = c^j$. Consider the following $z^N := \{(z^1(t), z^2(t))\}_{t \in T} \in F^N$:

	t_1^2	t_2^2
t_1^1	(-1, 1)	(1, -1)
t_2^1	(1, -1)	(-1, 1)

1.6 Descriptive *interim* solution concepts

Each player plays both the role of principal and the role of agent:
1. Players get together to make coordinated strategy choice as *principals*.
2. They decide on their self-sustaining strategy bundles (descriptive solution of the game).
3. Each player execute his agreed strategy as an *agent* in an *interim* period.
The solution is called *ex ante* (*interim*, resp.), if it is agreed upon in the *ex ante* period (in an *interim* period, resp.).

Endogenous determination of a mechanism.

DEF. (Wilson, 1978, +)
$\mathcal{S}$: a Bayesian society.
The private information case.
A strategy bundle $x^* \in X$ is called a *Bayesian incentive-compatible coarse strong equilibrium* of $\mathcal{S}$, if
(i) $x^* \in \hat{F}^N(x^*)$; and
(ii) it is not true that

$$\exists\ S \in \mathcal{N} : \left(\exists\ E \in \bigwedge_{j \in S} \left(\mathcal{T}^j \cap T(S)\right) : E \neq \emptyset\right) :$$
$$\exists\ x^S \in \hat{F}^S(x^*) :$$
$$\forall\ j \in S : \forall\ t \in E :$$
$$Eu^j(x^S, x^{*N \setminus S} \mid \mathcal{T}^j)(t) > Eu^j(x^* \mid \mathcal{T}^j)(t).$$

DEF. The *Bayesian incentive-compatible coarse core*, for the case:
F^S : constant correspondence.
u^j : depends only upon (c^j, t).

DEF. $\mathcal{S}$: a Bayesian society.
The private information case.
A strategy bundle $x^* \in X$ is called an *interim Bayesian incentive-compatible strong equilibrium* of $\mathcal{S}$, if
(i) $x^* \in \hat{F}^N(x^*)$; and
(ii) it is not true that

$$\exists\ S \in \mathcal{N} : \exists\ t^S \in T^S : \exists\ x^S \in \hat{F}^S(x^*) :$$
$$\forall\ j \in S :$$
$$Eu^j(x^S, x^{*N\setminus S} \mid t^j) > Eu^j(x^* \mid t^j).$$

DEF. The *interim Bayesian incentive-compatible core*, for the case:
F^S : constant correspondence.
u^j : depends only upon (c^j, t).

REMARK Formal similarity to Wilson's *fine core*, but quite different interpretation.

	fine core	*interim* core
stragegy	$\mathcal{T}^S$-meas.	private-meas.
interpret.	use of the full c.s.	not improve at any t

DEF. (Vohra 1999)
Bayesian incentive-compatible coarse core,
interim Bayesian incentive-compatible core,
for the Bayesian pure exchange economy:
Based on the mediator-based approach.
Essentially, they do not impose the private measurability.

2 Bayesian pure exchange economy

2.1 Bayesian incentive-compatible coarse core

PROP. (Ichiishi and Yamazaki, 2004)
$u^j(\cdot, t)$: continuous, concave, and weakly monotone in $\mathbf{R}^l_+$ for every $t \in T$.
$\Rightarrow$
There exists a Bayesian incentive-compatible coarse core net-trade plan.

REMARK Vohra's example of an empty Bayesian incentive-compatible coarse core (Vohra, 1999, example 3.2, pp. 136-138) is crucially based on his postulate that:
z^j : $\mathcal{T}^S$-measurable, rather than private-measurable.
Vohra's setup requires the presence of a mediator.

2.2 *Interim* Bayesian incentive-compatible core

Positive result for $l = 1$;
Negative result for $l \geq 2$.

2.3 Yannelis' negative result

EX. (A variant of Vohra, 1999, Ex. 2.1): Market for lemons with an empty Bayesian incentive-compatible *interim* core.

$$\begin{aligned} l &= 2, \\ N &= \{1, 2\}, \\ T^1 &= \{l, h\}, \\ T^2 &= \{t^2\}, \text{ (so } T \sim T^1) \end{aligned}$$

$$e^1(t) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e^2(t) \equiv \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

$$\begin{aligned} u^1(c, t) &:= \begin{cases} c_2 & \text{if } t = l \\ 10c_1 + c_2 & \text{if } t = h, \end{cases} \\ u^2(c, t) &:= \begin{cases} c_2 & \text{if } t = l \\ 15c_1 + c_2 & \text{if } t = h, \end{cases} \\ \pi^2(t) &= 1/2, \quad \text{for } t = l, h. \end{aligned}$$

2.4 Conditions for the existence

ASS. For each consumer j and each type t^j,

$$
\begin{aligned}
&\exists\ a^j(t^j) \geq \mathbf{0} : \exists\ b^j(t^j) \in \mathbf{R} : \forall\ c^j : \\
&\qquad Eu^j\left(c^j + e^j \mid t^j\right) = a^j(t^j)\left(c^j + e^j(t^j)\right) + b^j(t^j),
\end{aligned}
$$

DEF. the coalitionally feasible choice space,

$$
C_0^S := \left\{ c^S \in \mathbf{R}^{l\cdot|S|} \;\middle|\; \begin{array}{l} \forall\ j \in S : \forall\ t^j \in T^j : \\ c^j + e^j(t^j) \geq \mathbf{0} \\ \sum_{j\in S} c^j \leq \mathbf{0} \end{array} \right\}.
$$

Remark. $\mathbf{0} \in C_0^S$, (in particular, $C_0^S \neq \emptyset$).

DEF. (j, t^j) : an *agent*.
A : the set of all agents,

$$
A := \{(j, t^j) \mid j \in N, t^j \in T^j\}.
$$

$\mathcal{B}_0$: the family of all *admissible* blocking coalitions of agents,

$$
\mathcal{B}_0 := \left\{ B \subset A \;\middle|\; \begin{array}{l} [(i, t^i), (j, t^j) \in B, t^i \neq t^j] \\ \Rightarrow i \neq j \end{array} \right\}.
$$

Consumer-coalition S forms as a blocking coalition in $\mathcal{E}_{pe}$ at type profile $\bar{t}^S$, iff the admissible agent-coalition $B := \{(j, \bar{t}^j) \in A \mid j \in S\}$ forms.

For $B \in \mathcal{B}_0$, let

$$\begin{aligned} S(B) &:= \text{ the consumers represented by } B \\ t^j(B) &:= \ j\text{'s type for which } (j, t^j(B)) \in B \end{aligned}$$

.

DEF. the *maximal coalitional gain* for each $B \in \mathcal{B}_0$,

$$v(B) := \max_{c^S \in C_0^{S(B)}} \sum_{j \in S(B)} a^j(t^j(B))c^j.$$

Remark. The gain $v(B)$ depends upon $\{\underline{e}^j\}_{j \in S(B)}$, where $\underline{e}^j_h := \min\ e^j_h(t^j)$.

THEOREM (Ichiishi and Yamazaki, 2004) *$\mathcal{E}_{pe}$: satisfies the ASS.*
For all $\{\lambda_B\}_{B\in\mathcal{B}_0}$ ($\subset \mathbf{R}_+$) and all $\{\mu_j\}_{j\in N}$ ($\subset \mathbf{R}^l_+$) for which

$$
\begin{aligned}
&\forall\ i, j \in N : \\
&\sum_{B\in\mathcal{B}_0 : S(B)\ni i} \lambda_B a^i(t^i(B)) + \mu_i \\
&\quad = \sum_{B\in\mathcal{B}_0 : S(B)\ni j} \lambda_B a^j(t^j(B)) + \mu_j, \qquad (1)
\end{aligned}
$$

it follows that

$$\sum_{B\in\mathcal{B}_0} \lambda_B v(B) \ \le\ \sum_{j\in N} \mu_j \cdot \underline{e}^j. \qquad (2)$$

$\Rightarrow$

A Bayesian incentive-compatible interim *core net-trade plan of $\mathcal{E}_{pe}$ exists.*

On the condition [(1) $\Rightarrow$ (2)] of Result 2:

Consider for example a two-consumer economy ($N = \{1, 2\}$).

$v(B) = 0$ for all $B \in \mathcal{B}_0$
$\Rightarrow$
the condition in the theorem is automatically satisfied.

Otherwise, let

$$K^j := \text{ the cone generated by } \{a^j(t^j) \mid t^j \in T^j\}.$$

There exists nonzero $\{\lambda_B\}_{B \in \mathcal{B}_0}$ which gives rise to a member in $K^1 \cap K^2$
$\Rightarrow$
The condition of the theorem is violated (unless $v(B) \equiv 0$).

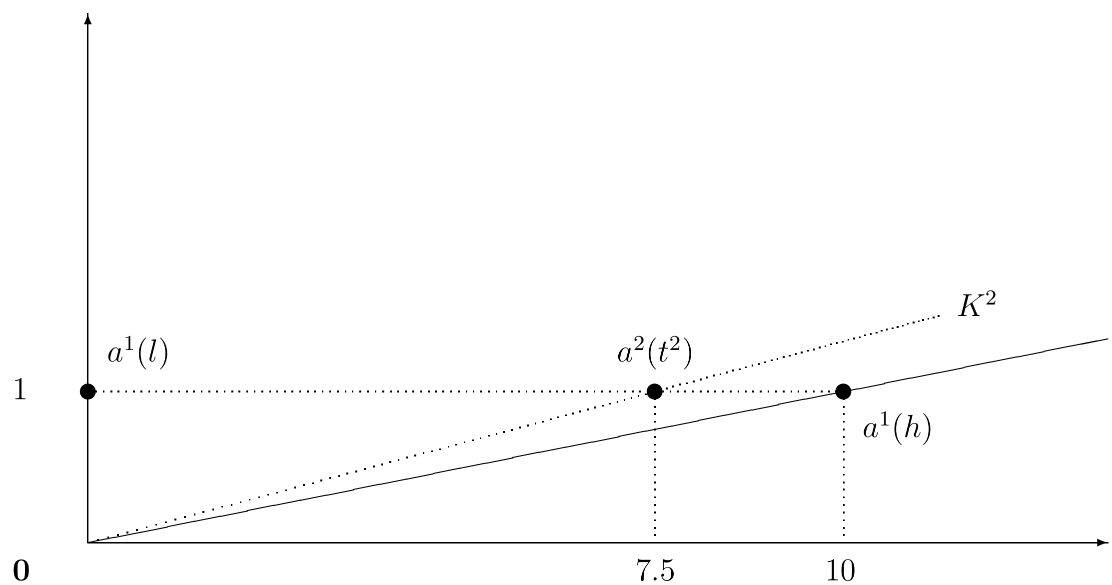

Figure 1: Market for lemons