Generalized Congestion Games

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Abstract

Basic definitions

A *Generalized Congestion Form*, or in short, *a G-congestion form* is a tuple $F = (M, N, (\Sigma_i)_{i\in N}, (w_a)_{a\in M})$ where

- M is a finite set of facilities (resources).
- N is a finite set of agents (players).
- $\Sigma_i \subseteq 2^M$ is the set of feasible subsets of 2^M (actions) for i.
  with the restrictions that $\Sigma_i \neq \emptyset$, and $\emptyset \notin \Sigma_i$.
- $w_a : L_N \to R$ is the facility-payoff function associated with $a \in M$, where

  $$L_N = \{(i, S) \in N \times 2^N | \quad i \in S\};$$

  If every agent j chooses $A_j \in \Sigma_j$, and for $A = (A_j)_{j\in N}$, $S_a(A) = \{j \in N | a \in A_j\}$, the total payoff of i is

$$u_i(A) = u_i(A_1, \cdots, A_n) = \sum_{a\in A_i} w_a(i, S_a(A)). \tag{1}$$

A *congestion form* is a G-congestion form for which for every facility $a \in M$ there exists $f_a : \{1, \cdots, n\} \to R$ such that

$$w_a(i, S) = f_a(|S|), \quad \text{for every } (i, S) \in L_N.$$

where $|S|$ denotes the number of agents in S. In between congestion forms and generalized congestion forms we define: A *congestion form with agent-specific payoffs* or in short an *AS-congestion form* is a G-congestion form such that for every $a \in M$ there exists a function $g_a : N \times \{1, \cdots, n\} \to R$ such that

$$w_a(i, S) = g_a(i, |S|), \quad \text{for every } (i, S) \in L_N.$$

Every G-congestion form F uniquely defines a game in strategic form, Γ_F, in which the set of players is N, Σ_i is the set of strategies of i, and the utility functions are given in (1). A game Γ in strategic form is called a *generalized congestion game*, or in short, *a G-congestion game* if $\Gamma = \Gamma_F$ for some G-congestion form F. The notions of *congestion games* and *AS-congestion games* are analogously defined.

An AS-congestion form is of type q if there exists $q|M|$ functions g^s_a : $\{1 \cdots, n\} \to R$,$a \in M$,$1 \leq s \leq q$ (not necessarily distinct), such that for every $i \in N$ there exists s such that for every $a \in M$, and for every $(i, S) \in L_N$, $w_a(i, S) = g^s_a(|S|)$. Obviously an AS-congestion form of type q is also of type

$q + 1$. Moreover, every AS-congestion form with n players is of type n. The *index* of an AS-congestion form, $q(F)$ is the minimal integer q for which it is of type q. Obviously $1 \leq q(F) \leq n$. Note that $q(F) = 1$ if and only if F is a congestion form.

The paper presents three types of results:

1. Characterization Results: Rosenthal (1973) proved that every congestion game is a potential game. Monderer and Shapley (1996) proved that every potential game is isomorphic to a congestion game. We generalize these theorems as follows:

Theorem 1 *Every AS-congestion form of type q induces a q-potential game.*

Theorem 2 *Every q-potential game is isomorphic to a AS-congestion game induced by an AS-congestion form of type q.*

An important corollary of Theorem 2 is:

Theorem 3 *Every game in strategic form is isomorphic to an AS-congestion game.*

2. Generalized Congestion Forms and Value Theory: In this section a *solution* is a function that assigns to every cooperative TU game $v : 2^N \rightarrow R$ a vector $\psi(v) \in R^N$.

Let $F = (M, N, (\Sigma_i)_{i \in N}, (w_a)_{a \in M})$ be a generalized congestion form. We say that F is defined by a solution ψ if there exists cooperative games v_a, $a \in M$ such that for every $i \in N$, and for every $S \subseteq N$ with $i \in S$, $w_a(i, S) = \psi v_a^S(i)$, where $v_a^S(T) = v_a(S \cap T)$ for every $T \subseteq N$. The following theorem generalizes a result in Monderer and Shapley (1996)

Theorem 4 *Let F be a generalized congestion form, and let ψ be an efficient solution. Γ_F is a congestion game if and only if ψ is the Shapley value on $\{v_a^S | \quad a \in M, S \subseteq N\}$.*

We also characterize semivalues.

3. Combinatorial auctions with strategic goods A valuation function defined on a set of goods N can be technically considered as a cooperative game on N. When the goods are, say workers, and there are buyers $b \in B$ who wish to purchase their services, then in a direct auction mechanism each such buyer submits a valuation function v^b. In view of previous results I analyze such auctions defined by a solution concept ψ as follows: Each buyer b submits v^b, every good chooses a buyer, and every buyer receives the set S_b of all $i \in N$ that choose b, and he pays to each $i \in S_b$, $\psi v_{S_b}^b(i)$.