Some variational convergence results for integral functionals using Young measures

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The study of variational convergence for integral functionals via the convergence of Young measures with applications to Control problems and some classes of evolution inclusions of second order was developed in a series of papers by Castaing-Raynaud de Fitte-Salvadori. Using the stable convergence and the fiber product for Young measures, we present here a more general study for integral functionals defined on $\mathrm{L}^1_{\mathbb{H}}([0,1],dt) \times \mathcal{Y}([0,1],\mathbb{Y})$ where $\mathbb{H}$ is a separable Hilbert space, $\mathbb{Y}$ is a Polish space and $\mathcal{Y}([0,1],\mathbb{Y})$ is the space of Young measures on $[0,1]\times\mathbb{Y}$. In particular, we consider two compact metric spaces $\mathbb{Y}$ and $\mathbb{Z}$ and the spaces of Young measures $\mathcal{H} = \mathcal{Y}([0,1],\mathbb{Y})$ and $\mathcal{R} = \mathcal{Y}([0,1],\mathbb{Z})$ and we prove that, under suitable hypotheses, the value function

$$U_J(\tau,x) = \sup_{\nu\in\mathcal{R}} \inf_{\lambda\in\mathcal{H}} \{ \int_\tau^1 [\int_{\mathbb{Z}} [\int_{\mathbb{Y}} J(t,u_{x,\lambda,\nu}(t),y,z)\,\lambda_t(dy)]\,\nu_t(dz)]\,dt \}$$

is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation $U_t(t,x) + H(t,x,\nabla U(t,x)) = 0$. Here $u_{x,\lambda,\nu}$ is the unique trajectory absolutely continuous solution of the evolution inclusion

$$\begin{cases} \dot{u}_{x,\lambda,\nu}(t) \in -\partial I_{\gamma,Y}(t,u_{x,\lambda,\nu}(t),\lambda_t) \\ + \int_{\mathbb{Z}}[\int_{\mathbb{Y}} g(t,u_{x,\lambda,\nu}(t),y,z)\,\lambda_t(dy)]\,\nu_t(dz), \;\; a.e. \quad \in [\tau,1], \\ u_{x,\lambda,\nu}(\tau) = x \in \mathbb{H}, \end{cases}$$

governed by the subdifferential operator $\partial I_{\gamma,Y}$ associated with the control Young measures $(\lambda,\nu)\in\mathcal{H}\times\mathcal{R}$, where J and g are Carathéodory integrands defined on $[0,1]\times\mathbb{H}\times\mathbb{Y}\times Z$, γ is a Carathéodory integrand defined on $[0,1]\times\mathbb{H}\times\mathbb{Y}$ that is convex on $\mathbb{H}$, $I_{\gamma,Y}$ is the convex integral functional defined on $[0,1]\times\mathbb{H}\times\mathcal{M}^1_+(\mathbb{Y})$ by $I_{\gamma,Y}(t,x,\nu) := \int_Y \gamma(t,x,y)\nu(dy)$ for all $(t,x,\nu)\in[0,1]\times\mathbb{H}\times\mathcal{M}^1_+(\mathbb{Y})$, and the integrand H in the HJB equation is

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defined on $[0,1]\times\mathbb{H}\times\mathbb{H}$ by

$$\begin{aligned} H(t,x,\rho) &= \inf_{\lambda\in\mathcal{M}^1_+(\mathbb{Y})}\sup_{\nu\in\mathcal{M}^1_+(\mathbb{Z})}\{\langle\rho,\int_{\mathbb{Z}}[\int_{\mathbb{Y}} g(t,x,y,z)\,\lambda(dy)]\,\nu(dz)\rangle \\ &\quad + \int_{\mathbb{Z}}[\int_{\mathbb{Y}} J(t,x,y,z)\,\lambda(dy)]\,\nu(dz)\rangle + \delta^*(\rho,-\partial I_{\gamma,\mathbb{Y}}(t,x,\lambda))\}. \end{aligned}$$

Some limiting properties for nonconvex integral functionals in proximal analysis are also investigated using Komlós convergence. In particular, the following holds. Let $\mathbb{H}$ be a separable Hilbert space. Let f be a nonnegative normal integrand defined on $[0,1]\times\mathbb{H}$. Let $(u^k)_{k\in\mathbb{N}\cup\{\infty\}}$ be a bounded sequence in $\mathrm{L}^\infty_{\mathbb{H}}([0,1])$ say, $N_\infty(u^k)\le R$ for all $k\in\mathbb{N}\cup\{\infty\}$ for some $R>0$, which converges to u^∞ for the norm N_∞, with $u^k\in\operatorname{dom} I_f$ for all $k\in\mathbb{N}\cup\{\infty\}$, $(\sigma^k)_{k\in\mathbb{N}}$ a bounded positive sequence in $\mathrm{L}^2_{\mathbb{R}}(([0,1])$ and $(\zeta^k)_{k\in\mathbb{N}}$ be a bounded sequence in $\mathrm{L}^1_{\mathbb{H}}([0,1])$. Assume that

$$f(t,v(t))\ge f(t,u^k(t))+\langle\zeta^k(t),v(t)-u^k(t)\rangle+\sigma^k(t)||v(t)-u^k(t)||^2$$

for all v in the closed ball $\overline{B}_{\mathrm{L}^\infty_{\mathbb{H}}([0,1])}(0,2R)$ and for all $t\in[0,1]$. Then there is a filter $\mathcal{F}$ finer than the Fréchet filter such that

$$\sigma(\mathrm{L}^\infty_{\mathbb{H}}([0,1])',\mathrm{L}^\infty_{\mathbb{H}}([0,1]) - \lim_{\mathcal{F}}\zeta^k = l\in\mathrm{L}^\infty_{\mathbb{H}}([0,1])'$$

and

$$\sigma(\mathrm{L}^2_{\mathbb{R}}([0,1]),L^2_{\mathbb{R}}([0,1]) - \lim_{\mathcal{F}}\sigma^k=\sigma\in\mathrm{L}^2_{\mathbb{R}}([0,1])$$

so that

$$\int_0^1 f(t,v(t))\,dt\ge\int_0^1 f(t,u^\infty(t))\,dt+\langle l,v-u^\infty\rangle+\int_0^1\sigma(t)||v(t)-u^\infty(t)||^2\,dt.$$

Consequently, $l\in\partial^p I_f(u^\infty)$. Further, let $l=l_a+l_s$ be the decomposition of l in absolutely continuous part l_a and singular part l_s, then we have

$$\begin{aligned}\int_0^1 f(t,v(t))\,dt\ge\int_0^1 f(t,u^\infty(t))\,dt+\int_0^1\langle l_a(t),v(t)-u^\infty(t)\rangle\,dt \\ +\int_0^1\sigma(t)||v(t)-u^\infty(t)||^2\,dt.\end{aligned}$$

In particular, assume that f is a lower semicontinuous function defined on $\mathbb{H}$, (u^k) pointwise converges to u^∞, and

$$f(x)\ge f(u^k(t))+\langle\zeta^k(t),x-u^k(t)\rangle+\sigma^k(t)||x-u^k(t)||^2$$

for all $k\in\mathbb{N}$, for all $x\in\overline{B}_{\mathbb{H}}(0,2R)$ and for all $t\in[0,1]$. Then, there is a Lebesgue-negligible set $\mathcal{N}$ such that for each $t\in[0,1]\setminus\mathcal{N}$ and each $x\in\overline{B}_{\mathbb{H}}(0,2R)$,

$$f(x)\ge f(u^\infty(t))+\langle\mathrm{bar}(\nu_t),x-u^\infty(t)\rangle+\sigma(t)||x-u^\infty(t)||^2,$$

where $\mathrm{bar}(\nu_t)$ denotes the barycenter of ν_t. In particular, the preceding inequality holds for each $x\in u^\infty(t)+\overline{B}_{\mathbb{H}}(0,R)$, in other words, $\mathrm{bar}(\nu_t)\in\partial^p f(u^\infty(t))$.

Further applications are presented in a forthcoming paper.