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Optimization and Lagrange multipliers: non-C^1 constraints and "minimal" constraint qualifications

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Abstract. Constrained optimization problems are central to economics, and Lagrange multipliers — when they exist — play a basic role in solving them, in theory and in practice. Examples are well known of optimization problems for which multipliers do not exist. So it is important to know what requirements constraint functions must satisfy to be "Lagrange regular," i.e. to guarantee existence of multipliers for broad classes of maximand or minimand functions. We relax the requirements in three directions:

1. We reduce the smoothness requirements on constraints. This allows weaker and more uniform hypotheses for mixed inequality and equality constraints, permitting, for example, just differentiability at the optimum and continuity in a neighborhood. (We allow much weaker hypotheses, as well.) Beyond smoothness, other requirements have long been imposed on constraint functions, to avoid simple examples lacking multipliers. We examine two types of such "constraint qualifications".

2. We provide new, relaxed constraint qualifications of both Jacobian and path types.

    (a) Our Jacobian constraint qualifications (23),(24),(25) permit spanning properties as alternatives to the usual rank restrictions.
    (b) Our path constraint qualifications (69),(72),(73) impose fewer restrictions than before on the directions permitted for constraint derivatives. (The logical relationships are indicated in (149)).

3. Our relaxed requirements are not only sufficient for avoiding many well-known counterexamples — they cannot be weakened further:

   (c) We formalize a notion of minimality for Jacobian constraint qualifications, and prove that ours are minimal for "Lagrange regularity."

   (d) We prove that our path constraint qualifications are necessary for "Lagrange regularity."

The tool enabling us to relax smoothness requirements on equality constraints is our Non-C^1 Implicit Function Theorem, p.142. (See also the reference in Section 8 to Halkin's work.)

Key words: Constrained optimization, Lagrange, Kuhn-Tucker, non-C^1 analysis, minimal constraint qualification, Jacobian Criterion, Tangency-Path Criterion